eliminating \( x, y, z \) the locus of these lines is the quadratic complex \( C \) with the equation \( \sigma \beta (Q_x^2 + Q_y^2) + Q_z Q'_z = 0 \).

Example 11. Show from Chapter II (6.6) that \( C \) is also the locus of the intersection of two consecutive homologous planes.

Example 12. If \( (x_i, y_i, z_i), i = 1, 2, \) are two distinct points on a line \( l \), show that \( l \) and its consecutive homologous line intersect if the points \( (x_1, y_1, z_1), (x_2, y_2, z_2), (x_1, y_1, z_1), (x_2, y_2, z_2) \) and \( (x_1, y_1, z_1), (x_2, y_2, z_2) \) are coplanar; show that the locus of the lines \( l \) with this property is also the complex \( C \), in accordance with the general two positions theory.

12. Another method to derive the screw displacement equations

So far we have used special coordinate systems in order to simplify our equations. Sometimes however it is desirable to express the screw transformation in terms of a general coordinate system. We have already given one form of the general rotation matrix in Chapter I, Example 4, and will develop it further in Chapter VI. Here we derive another useful (albeit, less elegant) form. Before deriving the general formulas we will develop a very useful equation known as Rodrigues' formula for a general screw displacement. We obtain the formula as follows: we take point \( P_1 \) and consider its displacement to position \( P_2 \) in \( S \). We consider its displacement as one whereby \( P_1 \) first rotates about the screw axis \( s \) to position \( P_1' \) and then translates parallel to \( s \) from \( P_1' \) to position \( P_2 \). It follows that \( P_1 \) and \( P_1' \) lie in the same plane normal to the screw axis. We take \( S_0 \) as the point where \( s \) cuts this plane, \( P_m \) the point where the perpendicular bisector of chord \( P_1P_1' \) cuts the chord, and the vectors \( r_1 \) and \( r_2 \) as respectively the position vectors of \( P_1 \) and \( P_1' \) relative to \( S_0 \). Then from Fig. 6 it follows that

\[
\tan(\phi/2) = P_m P_s / P_s S_p = |r_2 - r_1| / |r_1 + r_2|,
\]

where \( \phi \) is the rotation angle of the displacement about the screw axis. If we introduce the unit vector \( s \) along the screw axis (with its positive sense defined in accordance with the right-hand rule relative to the rotation angle \( \phi \)), we may write the vector expression

\[
r_2 - r_1 = (\tan(\phi/2))s \times (r_2 + r_1)
\]

which follows from (12.1) as far as magnitude is concerned, and in direction from the figure.

(12.2) is Rodrigues' formula for a planar displacement (Chapter VIII) with the origin at the center of the displacement.

If we consider the origin of coordinates located elsewhere on the screw axis say at point \( S_0 \) then the position vectors of \( P_1 \) and \( P_1' \) are \( R_1 \) and \( R_2 \), as shown in Fig. 7, and we have \( r_1 = R_1 - (S_0 S_p) s \), \( r_2 = R_2 - (S_0 S_p) s \) which when substituted into (12.2) yields

\[
R_2 - R_1 = (\tan(\phi/2))s \times (R_2 + R_1)
\]

which is Rodrigues' formula for a spherical displacement (Chapter VII) about an axis passing through the origin of coordinates.
If the origin of coordinates, O, does not lie on the screw axis, we take \( S_0 \) as the position vector from O to \( S_0 \), and \( P_1 \) and \( P_2' \) respectively as the position vectors of \( P_1 \) and \( P_2' \) from O. We have \( R_1 = P_1 - S_0 \), \( R_2 = P_2' - S_0 \) which when substituted into (12.3) yields

\[
P_2' - P_1 = (\tan (\phi/2)) s \times (P_2' + P_1 - 2S_0),
\]

which is Rodrigues' formula for general spherical displacement measured relative to any arbitrary origin O.

For a general spatial displacement we must add the screw translation, d, which carries \( P_2' \) to position \( P_2 \). Thus we substitute \( P_2' = P_2 - ds \) into (12.4), the result is

\[
P_2 - P_1 = (\tan (\phi/2)) s \times (P_2 + P_1 - 2S_0) + ds,
\]

which is Rodrigues' formula for a general screw displacement.

Formulas (12.2)–(12.5) were developed by Rodrigues [1840] and used by him to solve various problems dealing with the resultant of a series of displacements. Since they involve \( P_2 \) and \( P_1 \) on both sides of the equal sign, they usually require further manipulation. An alternative development starts with the fact that

\[
r_2 = r_1 \cos \phi + s \times r_1 \sin \phi,
\]

which follows from the planar figure (Fig. 6) if we drop the normal from \( P_2' \) to \( r_1 \), and keep in mind \( |r_1| = |r_2| \). Now substituting \( r_2 = R_2 - (R_2 \cdot s)s \), \( r_1 = R_1 - (R_1 \cdot s)s \) and using the facts that \( s \times r_1 = s \times R_1 \) and \( s \cdot R_1 = s \cdot R_2 \) yields

\[
R_2 = R_1 \cos \phi + s \times R_1 \sin \phi + (R_1 \cdot s) s(1 - \cos \phi).
\]

Substituting

\[
\cos \phi = (1 - \tan^2 (\phi/2))/(1 + \tan^2 (\phi/2)),
\]

\[
\sin \phi = 2(\tan (\phi/2))/(1 + \tan^2 (\phi/2))
\]

and rearranging (12.7) we get

\[
R_2 = R_1 + s \times R_1 \left[ 2(\tan (\phi/2))/(1 + \tan^2 (\phi/2)) \right]
+ \left[ 2\tan^2 (\phi/2)/(1 + \tan^2 (\phi/2)) \right] \left[ (R_1 \cdot s)s - R_1 \right]
\]

but since \( (R_1 \cdot s)s - R_1 = s \times (s \times R_1) \) we have

\[
R_2 = R_1 + \left[ 2(\tan (\phi/2))/(1 + \tan^2 (\phi/2)) \right] s
\times (R_1 + \tan (\phi/2)s \times R_1).
\]
and
\[ d_1 = ds_x - S_{o_x}(a_{11} - 1) - S_{o_y}a_{12} - S_{o_z}a_{13}, \]
\[ d_2 = ds_y - S_{o_x}a_{21} - S_{o_y}(a_{22} - 1) - S_{o_z}a_{23}, \]
\[ d_3 = ds_z - S_{o_x}a_{31} - S_{o_y}a_{32} - S_{o_z}(a_{33} - 1). \]

(12.13)

Hence (12.11) is a description of the displacement in terms of the screw parameters: \( \phi \) (the rotation angle), \( d \) (the translation distance), \( s \) (the unit direction along the screw), \( S_0 \) (a vector from the origin of coordinates to any point on the screw axis). At first it might seem as though we have eight rather than six parameters. However, \( s \) must satisfy \( s \cdot s = 1 \) and so it represents only two parameters. Similarly since \( S_0 \) is any point on the screw axis, \( S_0 \) only requires two parameters, hence one might set say \( S_{o_x} = 0 \) or, what is more usual, take \( S_0 \) as the foot of the normal from the origin, in which case \( S_0 \cdot s = 0 \). The signs of \( \phi, d \) are related to \( s \) by the right-hand-screw rule.

Example 14. Show that the \( a_i \) of (12.12) are the elements of an orthogonal matrix.

Example 15. Show that (12.12) and the form of \( A \) given in Chapter I, Example 4 are identical if \( c_0 = \cos \frac{\phi}{2}, c_1 = c_2 = c_3 = 0 \). This follows directly if we use half-angles in (12.12): \( a_{11} = 2(s_x^2(1 - \cos \frac{\phi}{2}) + 1) \), \( a_{12} = 2s_x s_y \sin \frac{\phi}{2} \), \( a_{13} = 2s_x s_z \sin \frac{\phi}{2} \), \( a_{22} = 2s_y \sin \frac{\phi}{2} \), \( a_{23} = 2s_y s_z \sin \frac{\phi}{2} \), \( a_{33} = 2s_z \sin \frac{\phi}{2} \).

Example 16. Show that if matrix \( A \) is given, the equivalent screw rotation can be determined from the relation
\[ \phi = \arccos\left(\frac{a_{11} + a_{22} + a_{33} - 1}{2}\right) \]
and the axis direction from
\[ s_x = \frac{(a_{23} - a_{32})}{(2\sin \phi)}, \quad s_y = \frac{(a_{31} - a_{13})}{(2\sin \phi)}, \quad s_z = \frac{(a_{12} - a_{21})}{(2\sin \phi)}. \]

Example 17. Show that (12.11) also follows from Rodrigues' formula (12.5). (Here, after substituting the scalar components, we must solve a system of three linear equations for \( X, Y, Z \), and then use \( \tan (\phi/2) = (1 - \cos \phi) / \sin \phi \) — the process is rather lengthy.)

Example 18. Show that the line with direction \( l \) (in both \( E \) and \( \Sigma \)) is displaced by the screw displacement into the line with direction \( l' \) in \( \Sigma \), as given by \( l' = Al \), with \( A \) defined by (12.12) or its equivalent.

Example 19. If we take an infinitesimally small displacement then \( \phi \rightarrow \Delta \phi, d \rightarrow \Delta d, P \rightarrow P + \Delta P, P' \rightarrow P' \), where \( A \) denotes a first order infinitesimal. Using this limiting process and then dividing by \( \Delta t \), show that Rodrigues' formula (12.5) yields
\[ \dot{P} = \omega \times (P - S_0) + \dot{S}_0, \quad \text{with} \quad \omega = \dot{\phi}. \]

Also show that (12.11) yields
\[ \dot{P} = BP + D, \]
where \( B \) is skew and has elements \( b_{11} = 0, b_{12} = -\omega_s x, b_{13} = \omega_s y, \) etc., and \( D \) has elements \( d_{11} = \dot{s}_x + \omega(S_y s_z - S_z s_y), \) etc.

\[ (12.14) \quad (P - Q) = (P - q) \times (P - Q) = (\tau/2)(P - q) \times (P - Q), \]

Equation (12.5) can be used to determine the screw parameters: Assume we have given two positions of \( E \) in terms of the positions of three non-collinear points say \( P, Q, R \). We write (12.5) twice: once for point \( P \) and once for point \( Q \). Subtracting the \( Q \) equation from the \( P \) equation yields
\[ (12.15) \quad [((R_2 - Q_2) - (P_2 - Q_1)) \times ((P_2 - Q_1) + (P_1 - Q_1))] \]

In obtaining (12.15) we have made use of the fact that \( [(R_2 - Q_2) - (R_1 - Q_1)] \) is perpendicular to \( s \), which is obvious if one substitutes \( R \) for \( P \) in (12.14).

If we operate on (12.5) with \( \cdot s \) and also set \( s \cdot s_0 = 0 \) we have for the normal, \( S_{o_n} \), to the screw from the origin,
\[ (12.16) \quad S_{o_n} = [(P_2 + P_1 + (s \times (P_2 - P_1))/\tan (\phi/2) - s \cdot (P_2 + P_1)] \]

Finally operating on (12.5) with \( s \cdot s_0 = 0 \) yields
\[ (12.17) \quad d = s \cdot (P_2 - P_1). \]

From (12.15) we obtain \( \phi \) and \( s \), from (12.16) \( S_{o_n} \), and from (12.17) \( d \). Hence the screw is completely determined from three non-collinear points.

Example 20. Show that if the displacement is such that (at least) one point of \( E \) remains fixed in \( \Sigma \) (i.e., it is either spherical (Chapter VII) or planar (Chapter VIII) we may set \( Q_1 = Q_2 = 0 \) in (12.15).

Example 21. Show that for the limiting case described in Example 19 formula (12.15) yields
\[ \omega s = [(\ddot{R} - \ddot{Q}) \times (\dot{P} - \dot{Q})]/(\ddot{R} - \ddot{Q}) \cdot (P - Q)), \]

Example 22. Consider the special cases for which (12.15) and the limiting case given in the previous example become indeterminate. (See for example Stielletts [1884].)

The theorem: a general displacement in three-space can be represented by a unique screw displacement is usually referred to as Chasles' theorem. (Although, both Mozi and Cauchy seem to have preceeded Chasles with this result.) We mention two important special cases: When \( d = 0 \) the displacement is a pure rotation. This causes no special difficulty, all the screw parameters are still unique and remain defined by the equations of this