Comments on mechanism kinematic chain isomorphism identification using adjacent matrices

J.P. Cubillo a,*, Jinbao Wan b

a School of Engineering and the Built Environment, University of Wolverhampton, Telford Campus, Shifnal Road, Wolverhampton TF2 9NT, UK
b Department of Mechanical and Electrical Engineering, Shenzhen Polytechnic, 518055 Shenzhen, Guangdong, People’s Republic of China

Received 16 September 2003; received in revised form 7 July 2004; accepted 7 July 2004
Available online 9 September 2004

Abstract

A number of theories on the relationship between adjacent matrices of isomorphic mechanism kinematic chains have been investigated. A particular published theory about mechanism kinematic chain isomorphism using adjacent matrices has been revised, after errors in the original theory were discovered. Subsequently, the necessary and sufficient conditions of the eigenvalues and eigenvectors of adjacent matrices for isomorphic kinematic chains have been proven rigorously. A new procedure to identify isomorphic chains has been developed and presented. With this new procedure it is only necessary to compare eigenvalues and several eigenvectors of adjacent matrices of isomorphic kinematic chains to identify the isomorphic chains. Some examples have been provided to demonstrate how to use the theory.

© 2004 Elsevier Ltd. All rights reserved.

1. Introduction

The study of kinematic structure is very important for designing new mechanisms. A major problem in this field is that of detecting possible isomorphisms between two given chains. An
undetected isomorphic chain results in duplicate solutions and unnecessary effort. A fault of isomorphic identification is the possible loss of candidates for new mechanisms. Numerous methods have been proposed to solve the problem, but some of them are too convoluted for engineers to use in practice. Characteristic polynomial based approaches [1,7] are less convoluted, but Mruthyunjaya [2,8] found that the approaches had failed to distinguish non-isomorphic chains since the indices of identification were necessary but not sufficient conditions for two chains to be isomorphic. Ambekar and Agrawal [9,10] introduced a method to compare a pair of chains for isomorphism by their codes of corresponding adjacency matrices. Although some improved and modified methods [3,11,12] have been published, they appear to be no more efficient. Rao and Varada Raju [13,14] introduced the concept of Hamming distances, from information and communication theory, to the study of kinematic structure. The proposed methods proved to be complex and cumbersome. Chu and Cao [4] presented a method which they named the adjacent-chain-table method. Quist and Soni [15] presented a method which utilizes the ‘loops’ of a chain for the identification of the chain. A method based on artificial neural network [5] theory was also investigated. It is clear that many new theories in other fields have been introduced in an effort to solve the problem of identifying isomorphic chains. However, published methods still leave a lot to be desired in different aspects, such as visual inspection, simplifying procedure of identification, adapting automatic computation and so on. Chang et al. [6] recently presented a new method based on eigenvalues and eigenvectors of adjacent matrices of chains. The method possesses the advantages of using standard matrix theory and adapting automatic computation techniques. However, the authors seem ambiguous about the key points of the method and include some fundamental errors in their theory. The purpose of this paper is to research the theoretic basis of this method and to demonstrate clearly a procedure to identify isomorphic chains.

2. Theory on kinematic chain isomorphism identification

Two chains are said to be isomorphic if there is a one-to-one correspondence between the links of one chain and those of the second chain such that two-links of a chain are jointed by a kinematic pair, or joint, if and only if the corresponding links of the other chain are jointed by a kinematic pair. Since a kinematic chain can be uniquely represented by a graph whose vertices correspond to the links of the chain and whose edges correspond to the joints of the chain, it means that the adjacent matrices of the graphs of two isomorphic chains can become equal by means of interchanging rows and column of one of them at same time. In other words, if \( A \) and \( A' \) are adjacent matrices of two isomorphic chains, there is a row transform matrix \( T \) such that

\[
TAT^{-1} = A'
\]  

(1)

where \( T \) is a product of the simple row transform matrix which interchanges two rows of \( A \) by left multiplication.

Chang et al. [6] presumes that the matrix \( T \) is a symmetrical and self-inverse matrix, i.e., \( T = T^T = T^{-1} \). This is a false premise. In fact, the matrix \( T \) is orthogonal only. The reason is that matrix multiplication is not commutative even if every multiplied simple row transform matrix is symmetrical and self-inverse.
From Eq. (1), adjacent matrices of two isomorphic chains are similar and have the same eigenvalues according to matrix theory.

Suppose $k_1, k_2, \ldots, k_n$ and $k'_1, k'_2, \ldots, k'_n$ are eigenvalues of adjacent matrices $A$ and $A'$ of any two kinematic chains with $n$ links, respectively, $x_1, x_2, \ldots, x_n$ and $x'_1, x'_2, \ldots, x'_n$ are eigenvectors corresponding to these eigenvalues. We construct two non-singular matrices $X$ and $X'$ called feature matrices of chains, with two groups of eigenvectors as column vectors.

$$X = \{x_1, x_2, \ldots, x_n\}$$

and

$$X' = \{x'_1, x'_2, \ldots, x'_n\}$$

Then

$$X^{-1}AX = \text{diag}\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$$

and

$$X'^{-1}A'X' = \text{diag}\{\lambda'_1, \lambda'_2, \ldots, \lambda'_n\}$$

If two kinematic chains are isomorphic, then the two groups of eigenvalues of $A$ and $A'$ are the same. Therefore, the diagonal matrix $\text{diag}\{\lambda'_1, \lambda'_2, \ldots, \lambda'_n\}$ can become equal to the diagonal matrix $\text{diag}\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ by interchanging a number of rows and columns. There exists a row transform matrix $T_c$ such that

$$X'^{-1}A'X' = \text{diag}\{\lambda'_1, \lambda'_2, \ldots, \lambda'_n\}$$

Let

$$X'_n = X'T_c$$

And therefore

$$X'^{-1}A'X'_n = \text{diag}\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$$

Chang et al. [6] considers that Eqs. (4) and (5) are necessary and sufficient conditions of isomorphism of kinematic chains. The following theorem and associated proof clearly show that these conclusions of Chang are wrong.

**Theorem.** Suppose that $X$ and $X'$ are feature matrices of kinematic chains with adjacent matrices $A$ and $A'$, respectively. Also, $\lambda_1, \lambda_2, \ldots, \lambda_n$ and $\lambda'_1, \lambda'_2, \ldots, \lambda'_n$ are eigenvalues corresponding to eigenvectors $X$ and $X'$. The necessary and sufficient conditions of isomorphism kinematic chains, that is there is a row transform matrix $T$ such that $TAT^{-1} = A'$, are

(a) there exists a row transform matrix $T_c$ such that Eqs. (4) and (5) are true,

(b) $TX$ is also a feature matrix of $A'$, that is

$$A'TX = TX \text{ diag}\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$$
Proof. The necessary conditions are that if \( A \) and \( A' \) are similar then they have the same eigenvalues which keep condition (a) true. Substituting Eq. (1) into (2), we get Eq. (6).

The sufficiency of conditions is also obvious. In fact, \( T A T^{-1} = A' \) can be obtained easily from Eqs. (6) and (2) directly.

The condition (b) is very important in identifying isomorphisms of kinematic chains. It can be shown that eigenvectors of an adjacent matrix of a chain can be transformed to that of another chain by some row transform matrix \( T \) when the chains are isomorphic. Especially, since an eigenvector corresponding to a single eigenvalue of a matrix is unique, except a coefficient, two transformed eigenvectors correspond to a single eigenvalue of \( A \) and \( A' \) of two chains are equal or pro-rata. It means only several eigenvectors need to be detected to determine isomorphic relationship corresponding links of chains or non-isomorphism of chains. Such characteristics can detect isomorphisms of kinematic chains conveniently and easily.

The condition given by Chang et al. [6] only indicates, in fact, that adjacent matrices of isomorphic chains have the same eigenvalues. Obviously, it is just a necessary but not sufficient condition for isomorphisms of chains. Chang et al. [6] did not find the most important characteristic represented by Eq. (6) of isomorphic chains. □

3. Procedure to identify isomorphism of kinematic chains

Based on the above theorem, it is clear that eigenvalues of adjacent matrices of chains are equal and eigenvectors of an adjacent matrix of a chain can be transformed to that of another chain by the same row transform matrix if chains are isomorphic, otherwise the chains are non-isomorphic. This prompted us to propose a procedure for identifying isomorphism or not as follows:

(1) Write out adjacent matrices of chains.
(2) Compute eigenvalues and corresponding eigenvectors of adjacent matrices.
(3) Compare with eigenvalues of adjacent matrices. If the eigenvalues of different adjacent matrices are correspondingly equal, then the next step should be done. Otherwise, the chains are non-isomorphic.
(4) Select several eigenvectors corresponding to a single eigenvalue from different adjacent matrices and compare with elements of eigenvectors to find the interchanging of rows. If the elements of eigenvectors corresponding to eigenvalues from different adjacent matrices are corresponding equal or pro rata, we make the interchanging of rows to one of the feature matrices and go-to the next step. Otherwise, the chains are non-isomorphic.
(5) Examine the interchanged feature matrix. If all of the vectors of the interchanged feature matrix are eigenvectors of a corresponding adjacent matrix, then the chains are isomorphic and we can indeed get the row transform matrix \( T \). Otherwise, the chains are non-isomorphic.

The above procedure can be executed automatically by a computer. To demonstrate more details and how to use the procedure more clearly, we give first a simple example of two kinematic chains with eight-links as shown in Fig. 1.
The adjacent matrices, noted down as $A$ and $B$, respectively, of the two chains are shown below:

$$
A = \begin{bmatrix}
0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 & 0
\end{bmatrix}, \quad
B = \begin{bmatrix}
0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \\
1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 0
\end{bmatrix}
$$

The eigenvalues and eigenvectors can be computed easily by the software MATLAB. The results are shown in Tables 1 and 2, in which the first row is eigenvalues.

From the tables, we can see that two adjacent matrices have the same eigenvalues. Select two eigenvectors corresponding to the first eigenvalue of adjacent matrices and compare with elements of the eigenvectors. We can find the correspondence between the links is given by $1 \leftrightarrow 6$, $2 \leftrightarrow 1$, $3 \leftrightarrow 5$, $4 \leftrightarrow 2$, $5 \leftrightarrow 4$, $6 \leftrightarrow 3$, $7 \leftrightarrow 7$, $8 \leftrightarrow 8$. So it is a pair of isomorphic.
chains. If we note $T(i,j)$ as a row transform matrix which interchanges the $i$th and $j$th row of $A$ when it is left multiplied by $A$, for example

$$ T(1, 6) = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} $$

Then

$$ T = T(1, 6)T(2, 6)T(3, 5)T(4, 6)T(5, 6) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} $$

Which makes $TAT^{-1} = B$.

It is obvious that all of $T(i,j)$ are symmetrical and self-inverse matrices but $T$ is not a symmetrical and self-inverse matrix.

Another example as shown in Fig. 2 is used to explain why we should select eigenvectors corresponding to single eigenvalue to find the interchanging of rows.

The adjacent matrices, noted down as $A$ and $B$ respectively, of the two chains are shown below:
Their eigenvalues and eigenvectors are shown in Tables 3 and 4, in which the eigenvalues \( \lambda_3 = \lambda_4 = -1.0 \) are complex. From the eigenvectors corresponding complex number, we cannot find the correspondence between the links even though two chains are isomorphic. The reason is that the linear combinations of eigenvectors corresponding a complex eigenvalue are also eigenvectors.

Select several eigenvectors corresponding to a single eigenvalue of adjacent matrices and compare with elements of the eigenvectors. We can find a correspondence between the links given by 1 \( \leftrightarrow \) 6, 2 \( \leftrightarrow \) 5, 3 \( \leftrightarrow \) 8, 4 \( \leftrightarrow \) 4, 5 \( \leftrightarrow \) 2, 6 \( \leftrightarrow \) 7, 7 \( \leftrightarrow \) 1, 8 \( \leftrightarrow \) 3. Then we can get the row transform matrix \( T = T(1, 6)T(5, 2)T(8, 3)T(6, 7) \), which makes \( B_1 = TA_1T^{-1} \). Also, the corresponding relationship of the links is not only for the chains given in Fig. 2 which are symmetrical.

An example, for which adjacent matrices of chains have the same eigenvalues, but the chains are non-isomorphic, is given by Chang et al. [6]. We have checked almost twenty eight-link chains.

<table>
<thead>
<tr>
<th>Table 3</th>
<th>Eigenvalues and eigenvectors of ( A_1 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-2.4142</td>
<td>-1.7321</td>
</tr>
<tr>
<td>0.5</td>
<td>0.0</td>
</tr>
<tr>
<td>-0.3536</td>
<td>-0.2299</td>
</tr>
<tr>
<td>0.3536</td>
<td>-0.2299</td>
</tr>
<tr>
<td>0.0</td>
<td>0.628</td>
</tr>
<tr>
<td>0.0</td>
<td>-0.628</td>
</tr>
<tr>
<td>-0.5</td>
<td>0.0</td>
</tr>
<tr>
<td>0.3536</td>
<td>0.2299</td>
</tr>
<tr>
<td>-0.3536</td>
<td>0.2299</td>
</tr>
</tbody>
</table>
to find such an example, but failed. It demonstrates that non-isomorphic chains can be identified faster than isomorphic chains by the method described in this paper.

4. Conclusion

Adjacent matrices of two isomorphic chains are similar and have the same eigenvalues. The similar relationship of adjacent matrices of two isomorphic chains is based on a special orthogonal row transform matrix. The special matrix can transform eigenvectors of an adjacent matrix of a chain to that of another chain when the two chains are isomorphic. These characteristics can be used easily to identify the isomorphism of chains and find the corresponding relationship of links of isomorphic chains and consequently the transform matrix.

The property of the transform matrix is demonstrated in this paper. The necessary and sufficient conditions of the eigenvalues and eigenvectors of adjacent matrices of isomorphic chains are rigorously proven. Hence the theory published by Chang et al. [6] about mechanism kinematic chain isomorphism using adjacent matrices has been revised.

The procedure, based on theoretic analysis, to identify isomorphism of kinematic chains has been verified and demonstrated by several examples. This method of identification of isomorphism of chains can exactly identify isomorphism of chains and be automatically executed by a computer. In particular, it can quickly check out non-isomorphic chains. It is the easiest and most convenient of the methods proposed thus far.

References