A new theory for the topological structure analysis of kinematic chains and its applications

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Abstract

Based on the array representation of loops in topological graphs of kinematic chains, this paper proposes two basic loop operations, “Θ” and “@”, for the first time. The existent conditions and properties of “@” operation are researched and four laws about the operation are presented. Furthermore, after the important concepts of the independent loop set and its selection theorem are proposed, the loop relationship of kinematic chains is revealed; thus an original theory of loop analysis is established. Finally, some applications are given under the basic theory above, such as the isomorphism identification, the detection of rigid sub-chains, and the freedom type analysis of kinematic chains.

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1. Introduction

The structural analysis and synthesis are two important aspects of the research on kinematic chains of mechanisms and robot manipulators. The structural synthesis usually creates a passel of kinematic chains, followed by isomorphism identification to eliminate duplicate kinematic chains and rigid sub-chain detection to eliminate wrong kinematic chains. Freedom and its types are very important in kinematic chain analysis.

Isomorphism identification among kinematic chains is a difficult problem [1,2]. There exist a wealth of literatures dealing with this topic. The characteristic-polynomial methods were studied by Uicker and Raicu [3], Yan and Hall [4,5], Mruthyunjaya [6–8], Mruthyunjaya and Balasubramanian [9], and Sohn and Freudenstein [10]. All these studies are based on adjacency matrix of kinematic chains or its different revisions. The main merit of these methods is that they are reasonably efficient in computation. But they are only the necessary condition for isomorphism identification and several counter-examples have been found [9]. The min-code method [11] requires a complicated analysis of distance matrix as well as the connectivity matrix to start its procedure. The Hamming-number approach [12–14] introduced a new idea for isomorphism identification

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and topological structure analysis. But when the primary Hamming string fails, the cumbersome computation of the secondary Hamming string is needed. The method based on eigenvector and eigenvalue [15–17] detects the isomorphism by computing both the eigenvectors and eigenvalues of vertex adjacency matrices. It is effective but not efficient enough. The genetic algorithm [18] and the artificial neural network approach [19] are also introduced to isomorphism identification, but their effectiveness needs further testing [20]. Although isomorphism problem has been studied for a long time, still more studies are desired.

This paper improves and develops the isomorphism research based on loop analysis of kinematic chains [21,22]. A more systematic research about loop is presented. New methods for isomorphism identification between kinematic chains, for rigid sub-chain detection in kinematic chains, and for the analysis of kinematic chain freedom type are also discussed respectively in the paper. With these methods a computer can easily fulfill these analysis tasks, thus saving human labor, and benefiting further automation and intelligence of the structural analysis and synthesis for mechanical conceptual design.

2. Basic theory

2.1. Basic concepts

A kinematic chain can be represented by a graph whose vertices correspond to links of the chain and whose edges to joints. The graph is called topological graph of the kinematic chain. For example, Fig. 1b is the topological graph of the kinematic chain in Fig. 1a.

A topological graph can be represented by an $n \times n$ matrix, where $n$ is the number of vertices. When vertex $i$ and vertex $j$ have been connected by an edge, the corresponding element $a_{ij}$ equals 1; otherwise, $a_{ij} = 0$. The matrix $A = [a_{ij}]_{n \times n}$ is the adjacency matrix of the topological graph. The adjacency matrix of Fig. 1 is

$$
A = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\
1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
\end{bmatrix}
$$

The number of edges connected with a vertex in a topological graph is defined as degree of the vertex. For example in Fig. 1b, there are three edges connected with vertex 4, so the degree of vertex 4 is 3. The degree of vertex can also been obtained from the adjacency matrix. The number of non-zero elements of the $i$th row or column is also the degree of the $i$th vertex.

In a topological graph, an open loop is a path consisting of an alternating sequence of vertex and edge, where no vertex or edge appears more than once and the starting vertex and ending vertex are different vertices.

![Fig. 1. An 8-bar kinematic chain and its topological graph.](image)
When the starting vertex and ending vertex in a path are the same, that is, every vertex in the path is connected with two edges, the path becomes a closed loop or a loop. As shown in Fig. 1, successive connecting vertices, 1, 2, 3, 4, 5, 6, 1, can form a loop.

Loop $i$ in a chain can be denoted by an $n$-dimensional array, $L(i)$, where $n$ is the number of vertices of the chain. For an $n$-dimensional array, $L(i)$, the $j$th element (the rightmost element is the first one and the leftmost element is the last one) indicates the relationship between vertex $j$ and loop $i$. If vertex $j$ exists in loop $i$, the $j$th element of $L(i)$ is “1”; otherwise, is “0”.

As shown in Fig. 1, $n = 8$, loop 1 only consists of vertices, 1, 2, 3, 4, 5, 6, 1, then loop 1 is expressed as

$$L(1) = [0, 0, 1, 1, 1, 1, 1, 1]$$

Note that, the rightmost element in $L(1)$ indicates the relationship between vertex 1 and the loop, the next rightmost element indicates the relationship between vertex 2 and the loop, and so on. It can be easily seen that vertices 7 and 8 are not in this loop.

Loop 2 denotes the loop consists of vertices, 1, 6, 5, 4, 7, 8, 1, hence

$$L(2) = [1, 1, 1, 1, 0, 0, 1]$$

Loop 3 denotes the loop consists of vertices, 1, 2, 3, 4, 7, 8, 1, hence

$$L(3) = [1, 1, 0, 0, 1, 1, 1]$$

Usually, in a kinematic chain, there are a lot of different loops. For example in Fig. 2, the kinematic chain has six different loops. In general, if a loop comprises loop $a_1$, loop $a_2$, ..., loop $a_k$, the loop can be expressed as $L(a_1 \oplus a_2 \oplus \ldots \oplus a_k)$ and is called a combinative loop. Note that in Fig. 1, loop 3 comprises loop 1 and loop 2. Loop 1 and loop 2 can be used to denote loop 3, that is

$$L(3) = L(1 \oplus 2)$$

The vertex number in a loop is defined as the size of the loop. When a loop is denoted by an $n$-dimensional array, the number of non-zero elements is its size. For example, there are six non-zero elements in $L(1)$, so the size of $L(1)$ is 6, that is

$$S[L(1)] = 6$$

2.2. Basic operations of loops

2.2.1. The “$\Theta$” operation of loops

The subtraction of every element in $L(b)$ from its corresponding element in $L(a)$ is defined as the “$\Theta$” operation of two loops $L(a)$ and $L(b)$, and denoted as $L(a) \Theta L(b)$. In each bit of the operation, if the difference is greater than 0, the result is “1”; otherwise, is “0”.

The result of $L(a) \Theta L(b)$ is also an $n$-dimensional array whose non-zero elements indicate the vertices exist in loop a but not in loop b.

The result of $L(a) \Theta L(b)$ is expressed as $P(a \Theta b)$. For example in Fig. 3

Fig. 2. An 8-bar kinematic chain and its topological graph.
The second bit of $P(1 \Theta 2)$ is not zero, which means vertex 2 is in loop 1 but not in loop 2. It is clear that $P(a \Theta b)$ no longer denotes a loop, and $N(P(1 \Theta 2))$ is used to denote the number of non-zero elements in $P(1 \Theta 2)$.

2.2.2. The “$\Theta$” operation of loops

Adding every element in $L(a)$ to its corresponding element in $L(b)$ is defined as the “$\Theta$” operation of two loops $L(a)$ and $L(b)$, and is expressed as $L(a) \oplus L(b)$. For each bit of the above operation, if the sum is bigger than zero but smaller than the local degree of the corresponding vertex, the result is “1”; otherwise, is “0”.

When the “$\Theta$” operation is involved, the local degrees of vertices are needed. They can be obtained by modifying the corresponding topological graph according to the following rules.

1. Remove all vertices that are not in either of the two operated loops and their corresponding connection relationships from the original topological graph.
2. Remove all inner connection relationships, if any, of the two loops from the original topological graph.

The vertex degree acquired from the modified topological graph is defined as the local degree of the vertex.

For example in Fig. 1, in the “$\Theta$” operation of $L(1)$ and $L(2)$, all vertices are either in loop 1 or in loop 2, and there is no inner connection relationship in loop 1 or loop 2, so the local degrees of vertices equal the original degrees of vertices. The “$\Theta$” operation of $L(1)$ and $L(2)$ can be expressed as follows.

$$L(1) = [0, 0, 1, 1, 1, 1, 1, 1]$$
$$L(2) = [1, 1, 1, 1, 0, 0, 1]$$
$$\oplus$$
$$P(1 \Theta 2) = [0, 0, 0, 0, 0, 0, 0, 1]$\

In the result of this operation, for the rightmost bit, the sum is 2. As the local degree of vertex 1 is 3, the result is 1; for the fifth bit from the rightmost, the sum is 2. As the local degree of vertex 5 is 2, the result is 0.

It can be seen that $L(1) \oplus L(2) = [1, 1, 0, 0, 1, 1, 1, 1]$ is the very combinative loop $L(1 \oplus 2)$. In a kinematic chain, if the combinative loop $L(a \oplus b)$ exists, it can be obtained through the “$\oplus$” operation of loop $a$ and loop $b$, that is

$$L(a \oplus b) = L(a) \oplus L(b)$$

For Fig. 2, in the “$\oplus$” operation of $L(1 \oplus 2)$ and $L(3)$ where
\[L(1 \oplus 2) = [1, 0, 1, 0, 1, 1, 1] \]
\[L(3) = [1, 0, 1, 1, 0, 0, 0] \]

vertex 7 is neither in loop \(L(1 \oplus 2)\) nor in loop 3, so vertex 7 and its corresponding connection relationship must be removed. So neither of the local degrees of vertices 1 and 3 is still 3. Both are 2.

For the sake of computer realization, the local degrees of vertices can also be obtained by modifying the corresponding adjacency matrix. For Fig. 2, the adjacency matrix is \(A\).

\[
A = \begin{bmatrix}
0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\
0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\
1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 \\
1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 1 \\
\end{bmatrix}
\]

In the “\(\oplus\)" operation of \(L(1 \oplus 2)\) and \(L(3)\), vertex 7 and its corresponding connection relationship need to be wiped away. That is, wipe off the 7th row and the 7th column from matrix \(A\). Then the resulting adjacency matrix is \(A_1\)

\[
A_1 = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\
\end{bmatrix}
\]

There are two non-zero elements corresponding to vertex 3 in matrix \(A_1\), so the local degree of vertex 3 is no longer 3 but 2. The local degree of vertex 1 is no longer 3 but 2. The local degrees of other vertices remain unchanged. Hence

\[L(1 \oplus 2) = [1, 0, 1, 0, 1, 1, 1] \]
\[L(3) = [1, 0, 1, 1, 0, 0, 0] \]
\[L(1 \oplus 2 \oplus 3) = [0, 0, 1, 1, 1, 1, 1] \]

For the chain in Fig. 3, in the “\(\oplus\)" operation of \(L(3)\) and \(L(4)\) where

\[L(3) = [1, 1, 1, 0, 0, 0, 1, 1, 1, 0] \]
\[L(4) = [1, 0, 0, 0, 1, 1, 0, 0, 1] \]

vertex 7 is neither in loop 3 nor in loop 4, so vertex 7 and its corresponding connection relationships must be wiped away. The local degree of vertex 8 is no longer 3, but 2, and the local degree of vertex 1 is no longer 4, but 3. The result can also be obtained by modifying the corresponding adjacency matrix. For example in Fig. 3, the adjacency matrix is \(B\)
In the “⊕” operation of $L(3)$ and $L(4)$, vertex 7 and its corresponding connection relationships need to be wiped away. That is, wipe off the 7th row and the 7th column of matrix $B$. The resulting adjacency matrix is $B_1$.

\[
B_1 = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\
1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\
\end{bmatrix}
\]

There are two non-zero elements corresponding to vertex 8 in matrix $B_1$, so the local degree of vertex 8 is no longer 3 but 2. The local degree of vertex 1 is no longer 4 but 3. The local degrees of other vertices are the same as their origin values. Hence

\[
L(3) = [1, 1, 1, 0, 0, 0, 1, 1, 1, 0] \\
L(4) = [1, 0, 0, 0, 1, 1, 0, 0, 1, 0] \\
L(3 ⊕ 4) = [1, 1, 1, 0, 1, 1, 1, 1, 1, 1]
\]

2.3. Conditions of the loop “⊕” operation

In general, the “⊕” operation of two loops forms a combinative loop. However, sometimes the result of $L(a) ⊕ L(b)$ cannot constitute a loop. In Fig. 2, the result of $L(1) ⊕ L(3)$ cannot constitute a loop. So in the “⊕” operation of loop a and loop b, in order to ensure that the result also forms a loop, the following conditions must be satisfied.

**Condition 1**  There are at least two shared vertices in the two loops. That is

\[
|N[L(a)] ∩ N[L(b)]| \geq 2
\]

**Condition 2**  The two corresponding bits of operated loops and their operation result are all “1”, which occurs exactly twice.

For example, in Fig. 2 where

\[
L(1) = [0, 1, 0, 0, 0, 1, 1, 1] \\
L(3) = [1, 0, 1, 1, 1, 0, 0, 0]
\]
there are no shared vertices in \( L(1) \) and \( L(3) \), so Condition 1 is not satisfied. The result of \( L(1) \oplus L(3) \) cannot form a loop, and is meaningless.

For Fig. 3 where

\[
L(1) = [0, 0, 1, 1, 0, 0, 0, 1, 1]
L(4) = [1, 0, 0, 0, 1, 1, 0, 0, 1]
\]

there is only one shared vertex in \( L(1) \) and \( L(4) \), so the result of \( L(1) \oplus L(4) \) cannot form a loop.

For Fig. 2, \( L(1) \oplus L(3) \) satisfies the existent condition of the “\( \oplus \)” operation, and the result is loop \( L(1) \oplus L(3) \), so

\[
L(1) \oplus L(3) = [0, 0, 1, 1, 1, 1, 1]
\]

Also for Fig. 2 where

\[
L(2) = [1, 1, 1, 0, 1, 0, 1, 1]
L(1 \oplus 2 \oplus 3) = [0, 0, 1, 1, 1, 1, 1]
\]

for four times, the two corresponding bits of operated loops and their operation result are all “1”. Nevertheless, Conditions 2 requires that such a case occurs exactly twice. So the result of \( L(2) \oplus L(1 \oplus 2 \oplus 3) \) cannot form a loop and is meaningless.

2.4. Properties of the loop “\( \oplus \)” operation

For the “\( \oplus \)” operation of any loops \( L(a), L(b) \) and \( L(c) \) in a kinematic chain, there are following properties.

1. **Commutative Law**

\[
L(a) \oplus L(b) = L(b) \oplus L(a)
\]

2. **Combination Law**

If both \( L(a) \oplus L(b) \) and \( L(b) \oplus L(c) \) satisfy the existent conditions of the “\( \oplus \)" operation of loops, then the “\( \oplus \)” operation satisfies the following combination law:

\[
L(a) \oplus L(b) \oplus L(c) = L(a) \oplus (L(b) \oplus L(c))
\]

3. **Self-Vanish Law**

\[
L(a) \oplus L(a) = \theta
\]

Here \( \theta \) denotes an \( n \)-dimensional array whose elements are all zero, and \( n \) is the dimension of array \( L(a) \).

**Proof.** When the “\( \oplus \)" operation is done on loop \( L(a) \) and itself, the operated corresponding bits are always the same. If both are 0, the result is 0; if both are 1, for the local degree of the vertex is 2, the result is also 0. So \( L(a) \oplus L(a) = \theta \).

4. **Absorption Law**

\[
L(a) \oplus L(a \oplus b) = L(b)
\]

**Proof.**

\[
L(a) \oplus L(a \oplus b) \equiv (L(a)\oplus (L(a)\oplus L(b)))\oplus L(b) \equiv (L(a)\oplus L(a))\oplus L(b) \equiv \theta \oplus L(b) = L(b)
\]
For example in Fig. 2

\[ L(1) = [0, 1, 0, 0, 1, 1, 1] \]
\[ L(1 \oplus 2) = [1, 0, 1, 0, 1, 1, 1] \]
\[ = [1, 1, 1, 0, 1, 1, 0, 1] \]
\[ = L(2) \]

That is

\[ L(1) \oplus L(1 \oplus 2) = L(2) \]

Other examples of the “⊕” operation in Fig. 2

\[ L(2) \oplus L(1 \oplus 2) = L(1) \]
\[ L(1) \oplus L(2 \oplus 3) = L(1 \oplus 2 \oplus 3) \]
\[ L(3) \oplus L(1 \oplus 2) = L(1 \oplus 2 \oplus 3) \]
\[ L(1 \oplus 2) \oplus L(1 \oplus 2 \oplus 3) = L(3) \]
\[ L(2 \oplus 3) \oplus L(1 \oplus 2 \oplus 3) = L(1) \]

But

\[ L(2) \oplus L(1 \oplus 2 \oplus 3) \neq L(1 \oplus 3) \]
\[ L(1 \oplus 2) \oplus L(2 \oplus 3) \neq L(1 \oplus 3) \]

The reason is that they do not satisfy the existent conditions of the “⊕” operation of loops and the results are meaningless.

3. Loop analysis

3.1. The independent loop set

Euler Theorem [23] pointed out that for a polygon meshwork with \( n \) vertices and \( m \) edges, the independent loop number \( v \) satisfies the following equation

\[ v = m - n + 1 \]

Obviously, Euler Theorem establishes the basic relationship among the vertex number, the edge number and the independent loop number of a topological graph. For the chain in Fig. 1, the independent loop number is \( v = 9 - 8 + 1 = 2 \).

If a loop set in a topological graph satisfies the following two conditions

(1) Its loop number is the same as the independent loop number \( v \) of a meshwork with as many vertices and edges, with \( v \) determined by the Euler Theorem.

(2) All other loops in this topological graph can be obtained through the “⊕” operation of those in the loop set; then the loop set is an independent loop set.

For example, in Fig. 1, for the loop set \( \{L(1), L(2)\} \), while its loop number is 2, the independent loop number \( v \) of a meshwork with as many vertices and edges is also \( 2(v = 9 - 8 + 1 = 2) \). The other loop satisfies the equation \( L(1 \oplus 2) = L(1) \oplus L(2) \). Thus both required conditions are satisfied. So the loop set \( \{L(1), L(2)\} \) is an independent loop set. It is obvious that loop set \( \{L(1), L(1 \oplus 2)\} \) or \( \{L(2), L(1 \oplus 2)\} \) can also serve as independent loop set.
3.2. The selection theorem for independent loop set

In general, for a kinematic chain, there are many different ways to select the independent loop set. The selection rule of independent loop set is given below.

**Theorem 1.** For the topological graph of a kinematic chain, select any loop set \( \{L(1), L(2), \ldots, L(v)\} \) consisting of \( v \) loops, with \( v \) determined by Euler Theorem. For any three different loops in the loop set, \( L(i), L(j) \) and \( L(k) \) \((i, j, k = 1, 2, \ldots, v)\), if there does not exist

\[
L(i) \oplus L(j) = L(k)
\]

the loop set \( \{L(1), L(2), \ldots, L(v)\} \) is an independent loop set.

For the chain in **Fig. 2**, there are six loops in all, \( L(1), L(2), L(3), L(1 \oplus 2), L(2 \oplus 3) \) and \( L(1 \oplus 2 \oplus 3) \). The independent loop number is \( v = 9 - 8 + 1 = 3 \). The independent loop set of this topological graph can be selected as \( \{L(1), L(2), L(3)\} \) or \( \{L(1), L(2), L(2 \oplus 3)\} \) or \( \{L(2), L(3), L(1 \oplus 2)\} \) or \( \{L(1), L(2), L(1 \oplus 2 \oplus 3)\} \) etc.

But neither \( \{L(1), L(2), L(1 \oplus 2)\} \) nor \( \{L(2), L(3), L(2 \oplus 3)\} \) can be selected as the independent loop set, because \( L(1) \oplus L(2) = L(1 \oplus 2) \) and \( L(2) \oplus L(3) = L(2 \oplus 3) \).

In other words, in a loop set if one loop can be obtained by using the “\( \oplus \)” operation of other loops in the same loop set, the loop set is not an independent loop set.

The usual selection rules for the independent loop set of a plane topological graph and non-plane one are given respectively below.

(1) For a plane topological graph, that is, a topological graph which can be represented by a plane graph, its mesh loops, within which there are no other loops, can be selected as the independent loop set. For example in **Fig. 3**, the mesh loop \( L(1) \) consisting of links 1, 2, 8 and 7 can be selected as one of the independent loops; the mesh loop \( L(2) \) consisting of links 1, 7, 8, 9 and 10 can be selected as another independent loop; the mesh loop \( L(3) \) consisting of links 1, 10, 4, 5 and 6 can be selected as a third independent loop; the mesh loop \( L(4) \) consisting of links 2, 3, 4, 10, 9 and 8 can be selected as a fourth independent loop. Obviously, the loop set \( \{L(1), L(2), L(3), L(4)\} \) satisfies the selection rules of independent loop set, so it is an independent loop set.

(2) For a non-plane graph, that is, a topological graph which cannot be represented by a plane graph, its mesh loops, if any, can also be selected as independent loops. The rule for selecting the rest independent loops is that every selected loop contains new vertices which other independent loops do not contain. For example, the topological graph for the kinematic chain shown in **Fig. 5 b** is a non-plane graph. The loop \( L(1) \) containing links 1, 2, 8 and 9 is one independent loop; the loop \( L(2) \) constituted by links 1, 7, 8 and 10 is another independent loop; the loop \( L(3) \) constituted by links 2, 3, 4, 8, 9 and 10 is a third independent loop; the loop \( L(4) \) consisting of links 5, 6, 7, 8, 9 and 10 is a fourth independent loop. Obviously, each of them contains a new vertex which other loops do not contain. So the loop set \( \{L(1), L(2), L(3), L(4)\} \) is an independent loop set.

3.3. The loop relationship of a kinematic chain

In a topological graph, when independent loops have been determined, the rest loops are defined as dependent loops. Note that the concept of dependent loops is different from that of combinative loops. A combinative loop is not necessarily a dependent loop and a non-combinative loop is not necessarily an independent loop. For example, in **Fig. 2**, when the three loops, \( L(1), L(2), \) and \( L(2 \oplus 3) \), are selected as independent loops, all other loops, \( L(3), L(1 \oplus 2), \) and \( L(1 \oplus 2 \oplus 3) \), are dependent loops. In this case, the loop \( L(2 \oplus 3) \) is at once a combinative loop and an independent loop, and the non-combinative loop \( L(3) \) is a dependent one.

For a kinematic chain, regardless of different drawing modes and different labeling modes, its loops depend only on its topological structure. We can summarize the above idea in the following theorem.
Theorem 2. For a topological graph, when a loop set is selected as its independent loop set, all other loops are dependent ones and can be obtained through the “⊕” operation of the independent loops.

For example in Fig. 2, there are six loops in all, \(L(1), L(2), L(3), L(4), L(5)\) and \(L(6)\), where \(L(4) = L(1 \oplus 2)\), \(L(5) = L(2 \oplus 3)\), and \(L(6) = L(1 \oplus 2 \oplus 3)\).

When a loop set \(\{L(1), L(3), L(6)\}\) is selected as the independent loop set, the rest loops, \(L(4), L(5),\) and \(L(2)\), are dependent loops and can be obtained by using the “⊕” operation of loops \(L(1), L(3)\) and \(L(6)\). That is

\[
\begin{align*}
L(2) &= L(1) \oplus L(6) \oplus L(3) \\
L(4) &= L(6) \oplus L(3) \\
L(5) &= L(1) \oplus L(6)
\end{align*}
\]

4. Application of loop theory

4.1. Isomorphism identification

The loop degree-sequence is the degree permutation of vertices sequenced one by one from a starting vertex along the loop clockwise or counterclockwise. Different starting vertices make different degree-sequences, and so do different directions. Those degree-sequences can be viewed as numbers. The largest number is defined as the canonical degree-sequence of the loop. For example in Fig. 1, the canonical degree-sequence of loop \(L(1)\) is 322322. And for Fig. 2, the canonical degree-sequence of loop \(L(6)\) is 332332. In a topological graph, the canonical degree-sequences of all its loops constitute a canonical degree-sequence set (CDSS) with its elements arranged from the smallest to the largest. In this case, the canonical degree-sequence set for a topological graph is unique.

For example in Fig. 1, the canonical degree-sequence set is

\[
\text{CDSS} = \{322322, 322322, 322322\}
\]

or shortened as \(\text{CDSS} = \{3 - 322322\}\).

For Fig. 4b, the canonical degree-sequence set is

\[
\text{CDSS}(b) = \{3322, 3322, 3333, 333322, 333322, 33223322\}
\]

or \(\text{CDSS}(b) = \{2 - 3322, 1 - 3333, 2 - 333322, 1 - 33223322\}\).

The canonical degree-sequence set of Fig. 4d is

\[
\text{CDSS}(d) = \{3232, 3232, 33332, 33332, 3323322, 3323322\}
\]

or \(\text{CDSS}(d) = \{3232, 3232, 2 - 33332, 2 - 332322\}\).

The canonical degree-sequence set of Fig. 4f is

\[
\text{CDSS}(f) = \{3232, 3232, 323322, 323322, 3232332, 3232332\}
\]

or \(\text{CDSS}(f) = \{2 - 3232, 4 - 332322\}\).

For a kinematic chain, when its independent loop set is determined, all other loops can be obtained through the “⊕” operation of these loops. It is clear that all loops in a kinematic chain depend only on its topological structure, regardless of different drawing modes and different labeling modes, so does the canonical degree-sequence set. As a result, we get a new isomorphism identification method of kinematic chains.

For two kinematic chains \(A\) and \(B\), if and only if their canonical degree-sequence sets are the same, that is

\[
\text{CDSS}(A) = \text{CDSS}(B)
\]

the two kinematic chains are isomorphic; if not, otherwise.
For example, none of the canonical degree-sequence sets are the same in Fig. 4. That is

\[ CDSS(b) \neq CDSS(d) \]
\[ CDSS(d) \neq CDSS(f) \]
\[ CDSS(f) \neq CDSS(b) \]

so the three kinematic chains are not isomorphism.

Another example, in Fig. 5, the independent loop set of (a) can be selected as \( \{L(1), L(2), L(3), L(4)\} \) with

\[ L(1) = [0, 1, 1, 0, 1, 0, 0, 0, 1, 1, 1, 1] \]
\[ L(2) = [1, 1, 1, 0, 1, 1, 0, 0, 0, 0, 0, 0] \]
\[ L(3) = [1, 1, 1, 0, 0, 0, 0, 0, 0, 0, 1, 1] \]
\[ L(4) = [1, 0, 0, 1, 1, 1, 0, 0, 0, 0, 0] \]

The canonical degree-sequence set of (a) is

\[ CDSS(a) = \{2 - 3333, 2 - 3332, 1 - 333333, 6 - 3333322, 1 - 33223322, 2 - 33323232\} \]

The independent loop set of (b) can be selected as \( \{L(1), L(2), L(3), L(4)\} \) with

\[ L(1) = [0, 1, 1, 0, 0, 0, 0, 1, 1, 1, 1, 1] \]
\[ L(2) = [1, 1, 1, 1, 0, 0, 0, 0, 1, 1, 1, 0] \]
\[ L(3) = [1, 1, 1, 1, 0, 0, 0, 1, 1, 1, 0] \]
\[ L(4) = [1, 1, 1, 1, 1, 0, 0, 0, 1, 1, 1, 0] \]

The canonical degree-sequence set of (b) is

\[ CDSS(b) = \{2 - 3333, 6 - 333332, 1 - 333333, 1 - 33223322, 2 - 33333322, 2 - 33323232\} \]

Note that

\[ CDSS(a) \neq CDSS(b) \]

so the two kinematic chains are non-isomorphism.
All 16 eight-bar single-DOF kinematic chains, 40 nine-bar two-DOF kinematic chains, 98 ten-bar three-DOF kinematic chains and 230 ten-bar single-DOF kinematic chains have been tested with this method and no counterexamples have been found.

4.2. Detection of rigid sub-chains

The rigid sub-chain inside a kinematic chain is a sub-chain with freedom equal to or less than zero, \( F \leq 0 \). Any link in a rigid sub-chain cannot move in relation to other links. Obviously, a closed sub-chain with only three links must be a rigid sub-chain. The number of the links in a rigid sub-chain, however, may be more than three, for example, the rigid sub-chain in Fig. 6. In the following, based on the loop set concept, the steps to detect rigid sub-chains in a kinematic chain are given.

Step 1 Determine the independent loop set of the kinematic chain. As mentioned above, for a given kinematic chain, there are many possibilities to choose its independent loop set. For convenience, choose the loop set whose loops are the smallest on the whole.

Step 2 Sequence all the elements in the independent loop set from the loop of the smallest size to the one of the largest. For example, for a chosen independent loop set \( \{ L(1), L(2), \ldots, L(v) \} \), the rearranged sequence is \( \{ L(1), L(2), \ldots, L(v) \} \) where 
\[
S[L(1)] \leq S[L(2)] \leq \ldots \leq S[L(v)]
\]

Step 3 If the size of loop \( L(1) \) is less than 4, \( S[L(1)] < 4 \), the loop is a rigid sub-chain; otherwise, go to Step 4.1.

Step 4.1 Find out a loop \( L_2(2) \). Choose a loop \( L(i) \) in the loop set residue \( \{ L(2), L(3), \ldots, L(v) \} \) as \( L_2(2) \), which satisfies that \( N[L(i) \Theta L(1)] \) is the smallest.
If two loops in this loop set satisfy this condition, choose the one of smaller size as \( L_2(2) \). But if both are the same size, either can be selected as \( L_2(2) \).

Step 4.2 If \( N[P_2(2)] \), where \( P_2(2) = [L_2(2) \Theta L(1)] \), is less than 2, the links in loops \( L_2(2) \) and \( L(1) \) form a rigid sub-chain; otherwise go to Step 5.1.
Step 5.1 Find out a loop $L_3(3)$. Choose a loop $L(i)$ in the loop set residue as $L_3(3)$, which satisfies that $N\{L(i)\Theta P_2(2)\Theta L(1)\}$ is the smallest. If two loops satisfy this condition, either can be selected as $L_3(3)$.

Step 5.2 If $N\{P_3(3)\}$, where $P_3(3) = L_3(3)\Theta P_2(2)\Theta L(1)$, is less than 2, the links in loops $L(1), L_2(2)$ and $L_3(3)$ form a rigid sub-chain; otherwise go to Step 6.1.

Step j.1 Find out a loop $L_{(j-2)}(j-2)$. Choose a loop $L(i)$ in the loop set residue as $L_{(j-2)}(j-2)$, which satisfies that $N\{L(i)\Theta P_{(j-3)}(j-3)\Theta \ldots \Theta P_2(2)\Theta L(1)\}$ is the smallest. If two loops satisfy this condition, either can be selected as $L_{(j-2)}(j-2)$.

Step j.2 If $N\{P_{(j-2)}(j-2)\}$, where $P_{(j-2)}(j-2) = L_{(j-2)}(j-2)\Theta P_{(j-3)}(j-3)\Theta \ldots \Theta P_2(2)\Theta L(1)$, is less than 2, the links in loops $L(1), L_2(2), \ldots L_{(j-2)}(j-2)$ form a rigid sub-chain; otherwise go to Step j + 1.1.

Step v+2 There is only one loop $L_{(v+2)}$ left in the loop set residue, where $v$ is the independent loop number. Suppose $P_v(v) = L_v(v)\Theta P_{(v-1)}(v-1)\Theta \ldots \Theta P_2(2)\Theta L(1)$

If $N\{P_{(v+1)}\}$ is less than 2, this kinematic chain contains a rigid sub-chain; otherwise there is no rigid sub-chain in the kinematic chain.

For example, in Fig. 6, $L(1)$ and $L(2)$, can be selected as two independent loops where

$L(1) = [0, \ldots, 0, 1, 1, 1, 1]$
$L(2) = [0, \ldots, 0, 1, 1, 1, 1, 0, 1]$

Here $S[L(1)] = S[L(2)] = 4$, the rearranged sequence can be $\{L(1), L(2)\}$.

Step 3 The size of loop $L(1)$ equals 4, so go to Step 4.

Step 4 There is only one loop $L(2)$ left. Let $L_2(2) = L(2)$. $N\{P_2(2)\}$ equals 1, where $P_2(2) = [L_2(2)\Theta L(1)] = L(2)\Theta L(1) = [0, \ldots, 0, 1, 0, 0, 0, 0, 0]$, so the links on loops $L(1)$ and $L(2)$ form a rigid sub-chain.

Take Fig. 5a as another example.

Step 1 the independent loop set can be selected as

$\{L^*(1), L^*(2), L^*(3), L^*(4)\}$

where

$L^*(1) = [0, 1, 0, 0, 0, 0, 1, 1, 1, 1]$
$L^*(2) = [1, 1, 0, 0, 0, 0, 1, 1, 0, 0, 0]$
$L^*(3) = [1, 1, 1, 0, 0, 0, 0, 0, 0, 1]$
$L^*(4) = [1, 0, 1, 1, 1, 1, 0, 0, 0, 0]$

Step 2 It is obvious that $S[L^*(2)] = S[L^*(3)] = 4$ and $S[L^*(1)] = S[L^*(4)] = 5$. The rearranged sequence can be as $\{L(1), L(2), L(3), L(4)\}$, where $L(1) = L^*(2), L(2) = L^*(3), L(3) = L^*(1), L(4) = L^*(4)$.

Step 3 The size of loop $L(1)$ equals 4, so go to Step 4.1.

Step 4.1 The chosen loop $L_2(2)$ is $L(2)$.

Step 4.2 $P_2(2) = L_2(2)\Theta L(1) = [0, 0, 1, 0, 0, 0, 0, 0, 0, 1, 1], N\{P_2(2)\} = 2$, so go to Step 5.1.

Step 5.1 The chosen loop $L_3(3)$ is $L(3)$.

Step 5.2 $P_3(3) = L_3(3)\Theta P_2(2)\Theta L(1) = [0, 0, 0, 0, 0, 0, 1, 1, 0], N\{P_3(3)\} = 2$, so go to Step 6.

Step 6 There is only one loop $L(4)$ left, so $L_4(4) = L(4)$. $P_4(4) = L_4(4)\Theta P_3(3)\Theta P_2(2)\Theta L(1) = [0, 0, 0, 1, 1, 0, 0, 0, 0, 0, 0], N\{P_4(4)\} = 2$, so there is no rigid sub-chain in this kinematic chain.

4.3. Detection of freedom types

Multiple degree-of-freedom ($F \geq 2$) kinematic chains have three types of freedom, the total freedom, partial freedom and fractionated freedom. The concepts of total and partial freedom of kinematic chains were originated by Hain [24] and the fractionated freedom was introduced by Monolescu [25]. The freedom types
of kinematic chains can also be determined by using the loop theory. In the following, the concepts of freedom types and their detection methods are introduced.

4.3.1. Total freedom
For a kinematic chain, if the mobility \( M \) of any loop is equal to or greater than \( F \) (\( F \) denotes the degree of freedom of the kinematic chain), that is \( M \geq F \), the kinematic chain has total freedom.

From the point of view of loops, if the smallest size of loops in a kinematic chain is equal to or greater than \( F + 3 \), that is

\[
\text{Min}\{S[L(1)], S[L(2)], \ldots, S[L(m)]\} \geq F + 3
\]

The kinematic chain is a chain with total-freedom.

For example, in Fig. 1, the smallest size of loops is 6, whereas the freedom of the kinematic chain is 3. So the kinematic chain has total-freedom.

4.3.2. Partial freedom
For a kinematic chain, if there exists at least one loop with the mobility \( M, 0 < M < F \), \( F \) denotes the degree of freedom of the kinematic chain), the kinematic chain is a chain with partial-freedom.

From the point of view of loops, if the smallest size of loops in a kinematic chain is greater than 3 but less than \( F + 3 \), that is

\[
3 < \text{Min}\{S[L(1)], S[L(2)], \ldots, S[L(m)]\} < F + 3
\]

The kinematic chain is a chain with partial freedom.

For the kinematic chain in Fig. 7, its degree of freedom is 2, and the smallest size of loops is 4, so the kinematic chain has partial-freedom.

4.3.3. Fractionated freedom
If a kinematic chain can be divided into two independent kinematic chains with a shared link or pair, and the sum of the degrees of freedom of the two independent parts is equal to the freedom of the whole chain, \( F \), the kinematic chain is said to be a chain with fractionated-freedom.

To put it in another way, the vertices of the topological graph can be divided into two sets with unique shared element. In other words, their intersection set has only one element. Moreover, all vertices of any loop of the topological graph exist in the same vertex set while the shared vertex exists in both sets at the same time.

For the kinematic chain in Fig. 8, its degree of freedom is 2 and its smallest size of loops is 4. The vertices of its topological graph can be divided into two sets

\[
\begin{align*}
\text{VS}_1 &= \{v_3, v_4, v_6, v_9\} \\
\text{VS}_2 &= \{v_1, v_2, v_5, v_7, v_8, v_6\}.
\end{align*}
\]

Moreover, the intersection set of \( \text{VS}_1 \) and \( \text{VS}_2 \) is \( \{v_6\} \), namely

\[
\text{VS}_1 \cap \text{VS}_2 = \{v_6\}
\]

Fig. 7. A 9-bar kinematic chain with partial freedom.
The kinematic chain has four loops, \( L(1) \), \( L(2) \), \( L(3) \) and \( L(2 \oplus 3) \).

\[
\begin{align*}
L(1) &= [1, 0, 0, 1, 0, 1, 0, 0] \\
L(2) &= [0, 1, 1, 1, 1, 0, 0, 1] \\
L(3) &= [0, 1, 0, 0, 1, 0, 0, 1] \\
L(2 \oplus 3) &= [0, 1, 1, 1, 0, 0, 0, 1]
\end{align*}
\]

Obviously, loop \( L(1) \) is constituted by the vertices in \( V_S1 \) and loops \( L(2) \), \( L(3) \) and \( L(2 \oplus 3) \) are constituted by the vertices in \( V_S2 \). So the kinematic chain is a chain with fractionated freedom.

5. Conclusions

From a new perspective, the paper proposes the original theory to analyze the topological structure of kinematic chains. First, two operations of loops, “\( \Theta \)” and “\( \oplus \)”, are introduced on the basis of the array representation of loops. Then the existent conditions and properties of “\( \oplus \)” operation are presented. Based on such important concepts on loops as the independent loop set and its selection rules, the theorem to analyze loop relationship of kinematic chains is established. Several applications, including the isomorphism identification, the detection of the rigid sub-chains, and the analysis of freedom types, are given. With the guidance of this new analysis theory, the analysis and synthesis of topological structure of kinematic chains can be easily carried out by a computer, which greatly benefits automation and intelligence of mechanical conceptual design.

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References