Some Applications of Graph Theory to the Structural Analysis of Mechanisms

Concepts in graph theory, which have been described elsewhere [2, 4, 6], have been applied to the development of (a) a computerized method for determining structural identity (isomorphism) between kinematic chains, (b) a method for the automatic sketching of the graph of a mechanism defined by its incidence matrix, and (c) the systematic enumeration of general, single-loop constrained spatial mechanisms. These developments, it is believed, demonstrate the feasibility of computer-aided techniques in the initial stages of the design of mechanical systems.

Introduction

The process of mechanical design involves intuition, experience, analysis, and synthesis. In recent times efforts have been made to develop an increasingly rational approach to engineering design. One of these involves the introduction of concepts of graph theory in the structural analysis of mechanical systems. In this sense, mechanisms and mechanical systems may be regarded as circuits with topological and network properties, just as electrical and other systems. The analysis of structure is intimately connected with type synthesis, that is to say, with the determination of the type of mechanism which is to synthesize the design requirements. In the present study the basic concepts of graph theory are applied to three general tasks, which need to be explored fully, for the efficient utilization of structural considerations. The first involves the comparison of one mechanical structure with another; the second, the automatic sketching of the structural components of a mechanism; and the third, the systematic structural classification of a particular class of mechanisms. Plane kinematic chains with turning pairs were selected to illustrate the first two tasks, and spatial mechanisms for the third.

Structural Comparison of Kinematic Chains and Mechanisms

Outline of Graph Theory

The following discussion refers to plane kinematic chains with turning pairs and binary joints; these will be abbreviated by the symbol BKC (basic kinematic chains). The results apply generally, however; that is to say, the techniques can be applied to plane and spatial kinematic chains with any types of joints. The structure of a kinematic chain may be defined by its kinematic graph, in which the links are represented by vertices, the joints by edges, and the edge connection between vertices corresponds to the joint coupling of links [6]. A mathematical representation of graphs, including kinematic graphs, uses matrices of incidence. Thus, the vertex-edge matrix of a graph is a rectangular matrix whose general element, $m_{ij}$, is 1 or 0 according as the $i$th edge is, or is not, incident at the $j$th vertex. When joints of several types are involved, each edge is assigned a "color" or distinguishing number defining the type of joint it represents.

The structure of a kinematic chain is completely defined by its kinematic graph, which in turn is defined by its vertex-edge matrix. Two vertex-edge matrices are equivalent, if they differ only by a permutation of rows and columns. In such cases the vertices and edges of one of the matrices can be relabeled so as to render the matrices identical. Two graphs are called isomorphic, if their vertex-edge matrices are equivalent. Hence basic kinematic chains are structurally identical if, and only if, their vertex-edge matrices are equivalent. Or in other words, a structural comparison between kinematic chains is reduced to the examination of the equivalence of their vertex-edge matrices. These facts are presented in greater detail in references [4 and 6].

Philosophies of a Computerized Structural Comparison

The vertex-edge matrix of a plane mechanism with turning joints, degree of freedom, $F$, and $L$ links is a rectangular matrix of rows and $1/4(8L - F - 3)$ columns. Hence the structural comparison of the BKC's of two such mechanisms by "brute force" involves a check of $(1/4)(8L - F - 3)!$ permutations, or relabeling operations of their vertex-edge matrices. Even for a modern computer, this number soon becomes astronomical and this procedure has to be abandoned. At the present time, no satisfactory algebraic formula is available which may be used to define structure in a manner independent of labeling operations. The trend, therefore, has been to develop "heuristic" programs using algorithms [5, 9]. These are programs which use successively more discriminating structural invariants of a graph (i.e., properties which are, for instance, independent of labeling operations but characteristic of graph structure) for comparison purposes; when a sufficient number of these are used in judicious combination and with programs "designed to avoid excessive computation along fruitless lines" [9], the structural comparison via programmed computation becomes economically feasible. These programs are called heuristic in the sense that there is no guarantee that sufficient structural discrimination is always available, i.e., it may be possible that in certain cases the program fails to detect differences in structure. In practice, however, this technique has been very successful. Several specialized computer languages have been developed (COMIT, SNOBOL) for use with structural or logical decisions.

Development of a General Computational Algorithm

The algorithm is illustrated with respect to the kinematic graphs, $G_1$ and $G_2$, of two basic kinematic chains. It involves the following sequence of steps:

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1. Check for equality the number of edges and vertices in $G_1$ and $G_2$. Unless equal, $G_1$ and $G_2$ are not isomorphic.

2. Check for equality the number of vertices of given degree in $G_1$ and $G_2$. If these are not correspondingly equal, $G_1$ and $G_2$ are not isomorphic.

3. Write an incidence table for each of the vertices of highest degree, defined as follows: The table has three rows; the first lists the edges incident at the given vertex; the second lists the other vertices at which these edges are incident; and the third lists the degrees of these vertices. If the degrees of these vertices are the same in two incidence tables, the corresponding vertices may denote the same link, or in computer language, they may be equated.

4. List all vertices which may be equated. For each of these, repeat step 3. Continue this procedure until, if possible, all vertices of $G_1$ are equated to the vertices of $G_2$. If this is not possible, the graphs are not isomorphic. If they are isomorphic, check the result by the indicated permutation of rows and columns.

For the two six-link BKCs, step 3 of the algorithm is illustrated in Fig. 1. The same procedure is applicable to mechanisms in which elements in the vertex-edge matrix associated with the fixed link are denoted by a special number, such as 2 for instance. Similarly, when several types of joints are involved, each type must be denoted by a distinguishing number. The structural analysis of basic kinematic chains is, in general, the most difficult, since it involves the least variation of structure in its elements.

The foregoing procedure, which is similar in spirit to that of Unger [9]—but shorter—has been applied successfully to ten-link P = 1 kinematic chains. The FORTRAN II program takes about 31/2 minutes on an IBM 7094 computer to complete a structural comparison of two chains. It is available from the first author upon request. Except in memory capacity and time, the program is not limited in the number of links it can accommodate. Furthermore, it is not heuristic in the sense that it should always succeed, but it is heuristic in the order in which it carries out this procedure. The significance of this program is believed to be in the demonstration of the feasibility of such techniques in the mechanisms field and in its generality, rather than its use in ten-link chains, in which structural identification can be carried out by hand.

Other Algorithms

Many structural properties of kinematic chains and of graphs are known [1–10]. All of these can be used. They may involve concepts of subgraph identification and loop formulas [8], contraction mapping [10], girth, diameter [2], and many others. Two "structural discriminants" which do not seem to have been used and which have been found very useful are the following:

(i) Suppose for each (independent and peripheral) loop of a planar graph of a kinematic chain, we list an ordered permutation of the vertices in the loop denoting a vertex of degree $r$ by the number $r$. The sequence may be clockwise or counterclockwise. The sense, however, should be identical for all loops except for the peripheral loop, for which it should be reversed. A sequence may begin with any vertex in the loop. Then the set of these permutation sequences seems to be a structural invariant of substantial discriminating power. For example, for the two six-link chains shown in Fig. 1, the sequences are:

Watt: (2323) (2332) (232323)
Stephenson: (2332) (233223) (2323)

and it is clear that these sequences represent structurally different chains.

(ii) Let the edges of the graph of a KKC be ordered into sets, each set consisting of all edges connecting vertices of the same degree. The number of edges in each set is a structural invariant. For the chains in Fig. 1, for instance, set 1 might consist of the joints connecting binary links, set 2 of joints connecting binary to ternary links, and set 3 of joints connecting ternary links. The Watt II program has 2, 4, 3 members in the three sets, respectively, whereas the corresponding numbers for the Stephenson chain are 1, 6, and 0.

Automatic Sketching of the Structural Elements of a Mechanism

The procedure is once again illustrated with respect to a KKC. The mathematical definition of the structure of the kinematic chain, which is fed into the computer, is the vertex-edge incidence matrix. It is required to produce an automatic sketch of the graph of the chain. The algorithm is again outlined by step:

1. Compute the degree of each vertex; this is equal to the sum of the entries in the row representing that vertex.

2. Beginning with a vertex of maximum degree, we begin to trace the first loop. Let the vertex associated with row $j$ be denoted by $v_j$. Find the first nonzero entry of row $j$, say $m_{jk}$. Then column $k$ is associated with edge $e_{jk}$ incident at vertex $v_j$.

3. Beginning with element $m_{jk}$ follow column $k$ to the second nonzero element, $m_{kp}$, say, corresponding to vertex $v_p$. We have now traced the elements $v_j - e_{jk} - v_p$ of the first loop.

4. Repeat steps 2 and 3 with vertex $v_p$, yielding, say, edge $e_{pq}$ and vertex $v_q$. We have now traced the elements $v_p - e_{pq} - v_q$ of the first loop.

5. Repeat steps 2, 3, 4 until the starting vertex, $v_j$, reappears. This completes the tracing of the first loop.

6. Set all entries which represent the edges of the loop just traced, equal to zero. This will prevent retracing of the same loop.

7. Repeat steps 2 to 6 with a vertex belonging to a loop already traced, and trace a second loop which will be completed when another vertex of a previously traced loop has been reached.

8. Repeat step 6 for the new loop.

9. Repeat steps 7 and 8 until all entries of the vertex-edge matrix are set to zero.

10. Assign coordinates to each vertex and provide instructions to connect these by edges when indicated.

11. Feed this information to an automatic curve-ploter. The result is the graph defined by its vertex-edge incidence matrix.

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The foregoing procedure is valid for both planar and nonplanar kinematic chains. In this initial study no attempt was made in step 11 to convert the graph to a line drawing of the mechanism, or to insure that planar graphs are drawn without crossing edges. These steps can also be developed. To differentiate a vertex from an edge crossing which is not a vertex, all vertices were numbered. The algorithm has been programmed in FORTRAN II for ten-link BKC's, and on the average takes about 17/4 minutes on an IBM 7094 computer. For sketching, the computer-output tape is fed into a Stromberg-Carson 40-20 plotter. The latter operation is a matter of seconds. Figs. 2 and 3 show the graphs of two ten-link kinematic chains, one planar and one nonplanar, obtained via the above algorithm.

The computer program is available from the first author upon request.

Structural Classification of Spatial Mechanisms

Introductory Comments

Spatial mechanisms with degree of freedom 1, and which obey the general freedom equations, possess one loop. We limit ourselves to the classification of such mechanisms with binary joints, whose elements involve surface contact. Table 1 lists these joints. The enumeration of these mechanisms represents an example of a technically significant class of mechanisms, whose classification follows directly from the concepts of graph theory via straightforward combinatorial methods. The enumeration includes by far the majority of spatial linkages known today.

No attempt is made here to include mechanisms with passive constraints or to exclude unworkable combinations, due to limitations which are not apparent from the graph alone. This is simply because of the need to start somewhere, rather than lack of appreciation of the latter. The present investigation follows earlier investigations by Harrisberger (7) and Roemer (8), on whose results the following development is based.

For constrained spatial mechanisms obeying the general freedom equations and for which the number of links is equal to the number of joints, it is readily shown that the summation, $f$, of the degrees of freedom permitted by the relative motion at each joint is equal to seven; furthermore, the number of links is less than or equal to seven.

Following Harrisberger, we enumerate by type and kind.

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Table 1  Kinematic joints whose elements involve surface contact

<table>
<thead>
<tr>
<th>Description</th>
<th>Type</th>
<th>Symbol</th>
<th>Rotation</th>
<th>Translation</th>
<th>Screw motion</th>
</tr>
</thead>
<tbody>
<tr>
<td>Thrust joint (revolute)</td>
<td>$b_1$</td>
<td>B</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Slicing joint (prismatic)</td>
<td>$p_1$</td>
<td>P</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>Helical joint (screw)</td>
<td>$p_2$</td>
<td>H</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>Cylindrical (cylindrical)</td>
<td>$p_3$</td>
<td>C</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>Slotted ball joint</td>
<td>$p_4$</td>
<td>E</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Spherical joint (ball joint)</td>
<td>$p_5$</td>
<td>S</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Plane joint</td>
<td>$p_6$</td>
<td>P</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

* 1 indicates freedom of relative motion permitted by the joint.

**Type.** Let $p_i$ denote the type of joint, the relative motion permitted by which has degree of freedom $i$. The type of a kinematic chain is the set of $p_i$ numbers associated with its joints. Hence, the number of types of the spatial kinematic chains considered here is equal to the number of partitions of the number 7 into $1's$, $2's$, and $3's$, corresponding to the character of the joints in Table 1:

**Type No.** | Nature of Joints
--- | ---
1 | $7p_1$
2 | $5p_1 + p_1$
3 | $4p_1 + p_1$
4 | $3p_1 + 2p_1$
5 | $2p_1 + p_1 + p_1$
6 | $p_1 + 3p_1$
7 | $p_1 + 2p_1$
8 | $2p_1 + p_1$

A type-5 kinematic chain, for instance, consists of two joints of degree of freedom 1, one joint of degree of freedom 2, and one joint of degree of freedom 3.
Kind. From Table 1, the $p_n$-joints are denoted by the symbols $R_i, P_i, H_i$; $p_{n}$-joints are denoted by the symbols $C_i, K_i$; $p_{n}$-joints are denoted by the symbols $S_i, D_i$. Hence for a given type of chain, we may identify the kinds of joints comprising the type by a permutation of the letters designating the joints. Type $2p_n + p_n$ chains, for instance, include the following kinds: $CCS, CCE, CCE, CCE$. These will be written in power index notation: $C S, C E, i.e., P^n$ denotes $n$-fold repetition of joint $P$.

The enumeration by type and kind was carried out by Harrisberger. We should like to add now the enumeration of the number of structurally distinct mechanisms for each type and kind. Some further subdivision is considered in [8], but without including subdivision by kind.

Enumeration of Spatial Mechanisms Via Graph Theory

Since we are dealing with single-loop kinematic chains, it is convenient to use the following definition concerning the joints of such a chain:

Definition. The permutation sequence of a single-loop spatial kinematic chain is defined by the ordered permutation of its joints, identified as in Table 1, starting with any joint and traversing the loop in either direction.

It is clear that in this definition, the permutation sequence and the graph are equivalent. For given the chain, we can always find the permutation sequence; and conversely, given the permutation sequence, we can always construct the graph.

Two permutation sequences are called equivalent if their graphs are isomorphic. In view of the above definitions, permutation sequences equivalent to a given permutation sequence can arise in three ways:

(a) By cyclic permutation (since the first element in the sequence is not defined),
(b) by an inverse permutation (since direction of the sequence is arbitrary),
(c) by permutation of repeated letters amongst themselves (since the permutation is not changed thereby).

The enumeration problem can now be stated as follows: Given a kinematic chain of specified type and kind, as defined by any permutation, $P^n$, of the letters designating the joints, find the number, $N$, of nonequivalent permutations of $P^n$.

We proceed as follows when $j$ is a prime number. The total number of permutations of $j$ objects is $j!$. To eliminate cyclic permutations divide $j!$ by $j$. To eliminate permutations of repeated letters amongst themselves, we must divide also by $j_1!j_2! \ldots j_l!$, where $j_1, j_2, \ldots, j_l$ denotes the number of times the $1st, 2nd, \ldots, lth$ joints are repeated. We need now to eliminate only inverse permutations, i.e.,

$$N = N' - \text{number of inverse permutations contained in } N''$$

To compute the number of inverse permutations, let us introduce the concept of a symmetric permutation. A symmetric permutation is defined as a permutation whose inverse is either the permutation itself or one of its cyclic permutations. For instance, PBREHRRP and CRRSR are such permutations. Intuitively the (colored) graph corresponding to such a permutation can be drawn so that it can have one axis of symmetry.

Let $s$ be the number of nonisomorphic symmetric permutations included in $N''$. Then the inverse permutations of these $s$-permutations are not included in $N''$. However, the inverse of each of the $(N'' - s)$ remaining permutations in $N''$ is included in $N''$. Hence the number of inverse permutations included in $N''$ is given by $1/2(N'' - s)$.

Therefore,

$$N = N' - 1/2(N'' - s) = 1/2 \left( j - 1 \right) + 1/2s$$

The number of symmetric permutations, $s$, is determined as follows: The permutations must be of the form $(am_k \ldots am_1 \ldots am_l)$ or $(am_1 \ldots am_k \ldots am_l)$ according as $j$ is even or odd; $s$ is equal to the number of nonidentical permutations of the elements $am_1 \ldots am_l$.

The number of mechanisms arising from each kinematic chain can be obtained by checking the number of distinct kinematic inversions for each chain. Let $P$ be the permutation which defines the given kinematic chain. The following cases arise:

(a) $P^n$ is symmetric and has an odd number of elements; the number of nonisomorphic mechanisms is $1/2 (j + 1)$, except in case (d).

(b) $P^n$ is symmetric and has an even number of elements; the number of nonisomorphic mechanisms is $1/2 j$, except in case (d).

(c) $P^n$ is not symmetric; the number of nonisomorphic mechanisms is $j$.

(d) All the joints of $P^n$ are identical; there is just one mechanism.

The result of the application of this analysis to the structural classification of single-loop spatial mechanisms (F = 1, joints as in Table 1, binary links) is shown in Tables 2 and 3.

It is possible also to derive (and thus check) these results by means of a more general method (Polya’s theorem; see, for instance, J. Riordan, Combinatorial Analysis, John Wiley & Sons, 1963), which, moreover, is applicable to other areas of mechanisms classification. We illustrate the method with reference to type $T_{3^n}$, kind $K_{6^n}$.

In the language of the theorem, the number of kinematic chains is equal to the number of inequivalent ways of coloring the edges of a plane regular septagon with three colors, three edges having the color “R,” two edges the color “P,” and two edges the color “H.” Two colorings are equivalent if the associated septagons can be brought into coincidence by a rigid-body motion. Equivalence, therefore, is defined by means of the rotational group of the regular septagon.

Table 2 Number of single-loop spatial kinematic chains (F = 1, joint types as in Table 1, binary links, when standard freedom equation is obeyed)

<table>
<thead>
<tr>
<th>$T_{3^n}$</th>
<th>$T_{2^n}$</th>
<th>$T_{1^n}$</th>
<th>$J_{6^n}$</th>
<th>$J_{5^n}$</th>
<th>$J_{4^n}$</th>
<th>$J_{3^n}$</th>
<th>$J_{2^n}$</th>
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<tr>
<td>$1^n$</td>
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<td>$5^n$</td>
<td>$6^n$</td>
<td>$7^n$</td>
<td>$8^n$</td>
<td>$9^n$</td>
</tr>
<tr>
<td>$10^n$</td>
<td>$11^n$</td>
<td>$12^n$</td>
<td>$13^n$</td>
<td>$14^n$</td>
<td>$15^n$</td>
<td>$16^n$</td>
<td>$17^n$</td>
<td>$18^n$</td>
</tr>
</tbody>
</table>

Table 3 Number of nonisomorphic symmetric permutations

<table>
<thead>
<tr>
<th>$T_{3^n}$</th>
<th>$T_{2^n}$</th>
<th>$T_{1^n}$</th>
<th>$J_{6^n}$</th>
<th>$J_{5^n}$</th>
<th>$J_{4^n}$</th>
<th>$J_{3^n}$</th>
<th>$J_{2^n}$</th>
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<td>$5^n$</td>
<td>$6^n$</td>
<td>$7^n$</td>
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<td>$13^n$</td>
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<td>$15^n$</td>
<td>$16^n$</td>
<td>$17^n$</td>
<td>$18^n$</td>
</tr>
</tbody>
</table>

2 If a permutation is considered as going from left to right, the inverse permutation goes from right to left.
### Table 2 (continued)

| A | B | C | D | E | F | A | B | C | D | E | F | A | B | C | D | E | F |
| $\text{Sp}_1^p \text{Sp}_2^p$ | $\text{Sp}_1^p \text{Sp}_2^p$ | $\text{Sp}_1^p \text{Sp}_2^p$ | $\text{Sp}_1^p \text{Sp}_2^p$ | $\text{Sp}_1^p \text{Sp}_2^p$ | $\text{Sp}_1^p \text{Sp}_2^p$ | $\text{Sp}_1^p \text{Sp}_2^p$ | $\text{Sp}_1^p \text{Sp}_2^p$ | $\text{Sp}_1^p \text{Sp}_2^p$ | $\text{Sp}_1^p \text{Sp}_2^p$ | $\text{Sp}_1^p \text{Sp}_2^p$ | $\text{Sp}_1^p \text{Sp}_2^p$ | $\text{Sp}_1^p \text{Sp}_2^p$ | $\text{Sp}_1^p \text{Sp}_2^p$ | $\text{Sp}_1^p \text{Sp}_2^p$ | $\text{Sp}_1^p \text{Sp}_2^p$ | $\text{Sp}_1^p \text{Sp}_2^p$ | $\text{Sp}_1^p \text{Sp}_2^p$ | $\text{Sp}_1^p \text{Sp}_2^p$ |
| $\text{Sp}_1^p \text{Sp}_2^p$ | $\text{Sp}_1^p \text{Sp}_2^p$ | $\text{Sp}_1^p \text{Sp}_2^p$ | $\text{Sp}_1^p \text{Sp}_2^p$ | $\text{Sp}_1^p \text{Sp}_2^p$ | $\text{Sp}_1^p \text{Sp}_2^p$ | $\text{Sp}_1^p \text{Sp}_2^p$ | $\text{Sp}_1^p \text{Sp}_2^p$ | $\text{Sp}_1^p \text{Sp}_2^p$ | $\text{Sp}_1^p \text{Sp}_2^p$ | $\text{Sp}_1^p \text{Sp}_2^p$ | $\text{Sp}_1^p \text{Sp}_2^p$ | $\text{Sp}_1^p \text{Sp}_2^p$ | $\text{Sp}_1^p \text{Sp}_2^p$ | $\text{Sp}_1^p \text{Sp}_2^p$ | $\text{Sp}_1^p \text{Sp}_2^p$ | $\text{Sp}_1^p \text{Sp}_2^p$ | $\text{Sp}_1^p \text{Sp}_2^p$ | $\text{Sp}_1^p \text{Sp}_2^p$ |

A: Type, B: Kind, C: No. of symmetrical chains, D: No. of chains, E: No. of non-symmetrical chains, F: No. of mechanisms.

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This group has the cycle index \( (\frac{1}{2}a)(a^2 + 2a + 6) \). Substituting \( t_1 = R + P + H, t_2 = R^2 + P^2 + H^2, t_3 = R^3 + P^3 + H^3 \) into the cycle index, the coefficient of the term \( R^2P^2H^2 \) will be found to be 18. According to Folya's theorem, this is the required number of kinematic chains, in agreement also with Tables 2 and 3.

Table 2 shows the existence of 188 kinds of kinematic chains, 776 structurally distinct kinematic chains, and 3862 structurally distinct mechanisms. Table 3 shows a breakdown of all the structurally distinct kinematic chains consisting of seven links. The number of distinct mechanisms seems noteworthy, especially when one considers that the input-output kinematics of each one is different. The classification of these mechanisms using permutation sequences leads to the thought that perhaps further developments are possible with approaches independent of labeling operations. This is expected to be the subject of future research.

Conclusion

Three representative problems in type synthesis, or structural analysis of mechanisms, have been considered from the viewpoint of graph theory. It is hoped that the results obtained so far may encourage further investigation along the lines of a computer-aided approach to the design of mechanical systems. For a more extended discussion of the theory on which the algorithms are based, the reader is referred to reference [4].

Acknowledgments

The authors would like to express their appreciation to the Army Research Office (Durham) for the support of this research; to Professors A. L. Dill Dare and C. W. McLarren, and Messrs. R. C. Baczoglini and D. M. Wallace, for discussions in connection with enumeration techniques; to Mrs. June H. Clearman for programming; and to Columbia University for the use of computational facilities.

References


Table 3: Enumeration of seven-link single-loop spatial kinematic chains (\( F = 1 \), joint types as in Table 1, binary links, when standard freedom equation is obeyed)

<table>
<thead>
<tr>
<th>KINEARMATIC CHAIN SPECIFICATION</th>
<th>327</th>
<th>328</th>
</tr>
</thead>
<tbody>
<tr>
<td>s^7</td>
<td>KKNK</td>
<td>KNNKN</td>
</tr>
<tr>
<td>s^6</td>
<td>KKKK</td>
<td>KKKK</td>
</tr>
<tr>
<td>s^5</td>
<td>KKKKK</td>
<td>KKKKK</td>
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<td>s^4</td>
<td>KKKKKK</td>
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<td>s^3</td>
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