An Application of Boolean Algebra to the Motion of Epicyclic Drives

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The kinematic structure of epicyclic drives has been investigated with the aid of Boolean algebra. The correspondence between the graph representation of the structure, the mechanism, and the form of the displacement equations has been derived. A canonical graph representation has been given. A method, believed to be novel, is described for the determination of the algebraic displacement equations by inspection, directly from the kinematic structure. The theory can be applied similarly to dynamic analysis and computer-aided sketching and animation.

Introduction

A number of recent studies [1-4] have been devoted to the network analysis of the kinematic structure of mechanisms. The design of mechanisms may be envisioned as beginning with such a structural analysis, followed by kinematic, dynamic, and further analysis.

In this investigation we have considered the elements of such a development in the case of epicyclic drives; in part, because their kinematic structure has been studied previously; in part, because they may be considered representative of linear mechanical systems; and in part because the problem is soluble.

The correspondence between the graph representation, the mechanism, and the form of the displacement equations naturally leads to propositional statements, which can be evaluated in terms of a modulo-2, or Boolean-ring algebra. As a result, the finite form of the displacement equations are obtained directly from the graph by purely formal operations requiring no sketches, visualization, or ad hoc procedures. The same applies to the dynamical equations and the sketching and animation of the mechanism.

The investigation most near in spirit to the present is J. W. Polder [12], whose approach, however, is based on a network analysis of a different nature. Epicyclic drives have been the subject of many investigations extending over a long period of time (e.g., [6-10, 12, 13] and others too numerous to mention). This investigation represents a new look at an old problem.

The Graph of Epicyclic Drives

As shown previously [3, 4], it is useful to define the graph of a mechanism as a linear graph in which the links correspond to vertices, the kinematic pairs to edges, and the pair-connection between links corresponds to the edge-connection between vertices. Edges are labeled according to the type of pairing they represent. The vertex representing the fixed link is labeled likewise. We limit ourselves in what follows to mechanisms which obey the general degree-of-freedom equation, the graphs of which are planar and the joints of which are binary.

For the graphs of geared kinematic chains representing epicyclic drives, it has been shown [1, 2] that:

G1 The number of vertices, \( v \), exceeds by unity the number of edges representing turning pairs.
G2 The number of geared edges (\( e_g \)), the number of vertices, and the degree of freedom of the mechanism \( F \), are related by the equation:
\[
e_g = v - 1 - F.
\]
G3 The subgraph obtained by deleting the geared edges is a tree.
G4 Each fundamental circuit associated with this tree (\( f \)-circuit for short) contains a geared edge as chord.
G5 The number of \( f \)-circuits is equal to the number of gear pairs.
G6 Each turning-pair edge can be characterized by its level, which identifies the location of its pair axis in space.
G7 The differential degree of freedom of any circuit must be at least 1; for an \( f \)-circuit, it is equal to the number of vertices in the circuit diminished by 2.
G8 In each \( f \)-circuit there is one vertex, called the transfer vertex, such that all edges on one side of the transfer vertex are at the same level and edges on opposite sides of the transfer vertex are at different levels. Without loss of generality we may restrict ourselves to graphs for which edges having a common level are incident at a common vertex. And any vertex of degree 2 having incidence only with turning-pair edges is a transfer vertex.

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Numbers in brackets designate References at end of paper.

Determination of the Transfer Vertices

Assuming a graph satisfies conditions G1 through G7, we can proceed as follows to determine the transfer vertices in accordance with G8, or prove their nonexistence.

Let $i, j, k$ denote 3 vertices in one $f$-circuit, which are connected by turning-pair edges $ij$ and $jk$. The pair of edges $(ij, jk)$ are called an adjacent edge pair. Assign the function $x_{ijk}$ to the edge pair $(ij, jk)$, where $x_{ijk}$ is equal to 1 or 0, according as vertex $j$ is, or is not, a transfer vertex in the circuit.

Further, in accordance with the notation of propositional calculus, let

+ denote the “exclusive or” (it is more usual to denote logical addition by the plus sign and some writers then use 0 for the exclusive or; in this case, however, the determinantal solution of the logical equations suggests the present choice of notation).
- (product symbol) denote the “and” function.
1 denote the truth of a logical proposition.
0 denote the falseness of a logical proposition.
= denote logical equivalence.

Then the existence of a transfer vertex in any $f$-circuit is equivalent to the logical condition that the functions $x_{ijk}$ are nonzero for exactly one edge pair in the circuit.

A necessary condition for this to be the case is the logical equation:

$$x_{ijk} + x_{ijk} + \ldots + x_{pqr} = 1 \quad (1)$$

where the left-hand side includes all edge pairs in the $f$-circuit. There are $(n-1-F)$ such equations. Equation (1), however, is not sufficient. For it does not exclude the possibility that the number of nonzero values of the functions $x_{ijk}$ in any given $f$-circuit is an odd integer greater than unity. Hence, we add the logical restriction:

$$x_{ijk}x_{pqr} = 0 \quad (2)$$

where the functions $x_{ijk}, x_{pqr}$ are any three functions associated with any 3 edge pairs in an $f$-circuit.

An additional restriction can arise when a vertex lies in more than two $f$-circuits. Suppose there are $k$ turning-pair edges incident at the vertex, where $k > 2$. The levels of the $k$ edges determine the values of the $(1/2)k(k-1)$ functions of the corresponding edge pairs. For example, let $(ij, (ik), \ldots$ be 3 edges incident at vertex $i$. Then the value of the functions of any two of the edge pairs determines the value of the function associated with the third pair in accordance with the following logical equations:

$$x_{ijk} + x_{ikl} + x_{iij} = 0 \quad \text{or} \quad x_{ijk} = x_{ikl} = x_{iij} = 1 \quad (3a, b)$$

There are $(1/2)(k-1)(k-2)$ independent equations of the type $(3a, b)$ at the vertex. However, we need equation $(3a, b)$ only for turning-edge pairs, which lie in an $f$-circuit.

The systems of equations (1), (2), (3) are called Boolean equations and obey the algebra of the residue class modulo 2.

Equations (1), (3) can be solved via Cramer’s rule in the field modulo 2, with the conditions for solvability and uniqueness in terms of rank, as in ordinary, linear, simultaneous equations. Any solutions which do not satisfy equation (2) are rejected. The solution and uniqueness of linear Boolean ring equations is discussed by Parker and Bernstein [11]. Mathematical operations in modulo-2 algebra are discussed, for example by Seshu and Reed [14]. For $m$ equations in $n$ unknowns, Parker and Bernstein [11] show that:

1. When $n = m$, a necessary and sufficient condition for the uniqueness of the solution is that the determinant $\nabla$ of the coefficient matrix be equal to unity.
2. When $n > m$, the solution is not unique.

Case 1 occurs in the majority of single-degree-of-freedom epicyclic drives. When $n > m$, the number of solutions cannot exceed $2^{m-n}$. If $\nabla = 0$, the equations are dependent or incompatible. If a solution to equations (1), (3) does not satisfy equation (2), the solution is inadmissible and the graph is not that of an epicyclic drive.

Example. A kinematic chain from which an epicyclic drive can be obtained by keeping one link fixed, is called a geared kinematic chain or an epicyclic chain. Fig. 1 shows the graph of an epicyclic chain with 2 $f$-circuits, I and II. The graph of an epicyclic drive is obtained by labeling the vertex representing the fixed link.

For circuit I, equation (1) becomes: $x_{123} = 1$; for circuit II, equation (1) becomes: $x_{123} + x_{234} = 1$. Equations (2) and (3) are not applicable.

In this case $\nabla = 1, x_{123} = 1$, and $x_{234} = 0$. Vertex 2 is the transfer vertex in both circuits.

Determination of the Edge Levels

The edge levels are governed by condition G8. In some cases more than one edge-level distribution may correspond to a given set of transfer vertices.

We can order the edges into sets, each set corresponding to one level, and determine edge levels by the operations of set inclusion and set exclusion, in accordance with condition G8. Two edge-level distributions are distinct or identical according to the nonexistence or existence of a permutation of the vertex-induced group of automorphisms of the graph, which transforms one distribution into the other.

Example. For the graph of the epicyclic chain shown in Fig. 2, vertex 1 is the transfer vertex in each $f$-circuit. There are 4 edge-level distributions as follows:

$$(12)(13)(14)(15)$$
$$(12)(14)(13, 15)$$
$$(13)(14)(12, 15)$$
$$(12, 14)(13, 15)$$
Rotational Displacement Equations

Let \( i, j \) denote the vertices of the geared edges of an \( f \)-circuit in which the transfer vertex is vertex \( k \); let \( \omega_i, \omega_j, \omega_k \) denote the angular velocities of links \( i, j, k \) (we shall always assume that vertex \( p \) represents link \( p \)) and let \( N_{ji} \) denote the gear ratio of the gear pair \( j, k \), positive or negative according as the mesh is internal or external (\( |N_{ji}| = T_j/T_k \), where \( T_j, T_k \) denote the number of teeth on gears \( j, k \)).

The constancy of the center distance between meshing gears is maintained by the pair axes of the link represented by the transfer vertex. This link will be called the gear carrier. Hence, links \( i, j, k \) constitute a simple epicyclic train for which:

\[
\omega_i - \omega_k = N_{ji}
\]

or

\[
\omega_i - \omega_j N_{ji} + \omega_k (N_{ji} - 1) = 0
\]  

(4)

The equation is well-known as the basic equation for the single epicyclic drive (e.g., Merritt [10]). The sum of the coefficients of the angular velocities is equal to zero.

There are \( (n-1)F \) equations of type (4), one for each \( f \)-circuit. Together they determine the rotational displacements of the epicyclic chain.

Example. We choose for example the epicyclic chain shown in Fig. 3(a); the graph is shown in Fig. 3(b). Vertex 1 is the transfer vertex for the three \( f \)-circuits. Hence, there are 3 rotational displacement equations:

\[
\begin{align*}
\omega_1 - \omega_2 N_{21} + \omega_2 (N_{21} - 1) &= 0 \\
\omega_1 - \omega_3 N_{31} + \omega_3 (N_{31} - 1) &= 0 \\
\omega_1 - \omega_4 N_{41} + \omega_4 (N_{41} - 1) &= 0
\end{align*}
\]  

(4a)

If link 4 is fixed (\( \omega_4 = 0 \)), we obtain the coupled epicyclic drive illustrated by Merritt [10, p. 156].

We note that:

1. The equations are determined by inspection from the kinematic structure in a purely formal manner.

2. It is not necessary to draw the graph; all that is needed is to identify the gear carrier (transfer vertex) for each gear mesh.

3. The equations pertain to the epicyclic chain and are valid for all mechanisms (epicyclic drives) derived from the chain.

4. The procedure can be used also in multi-degree-of-freedom epicyclic drives.

5. The example can be solved by other methods, e.g., that of Merritt [10], which is of a more indirect nature.

6. The equations can be used also in power-flow calculations.

Linear Displacement Equations

Let

\( j, k \) = vertices incident at geared edge \( jk \)

\( l \) = transfer vertex in \( f \)-circuit containing edge \( jk \)

\( a_{jk} \) = vector distance between axes of the gears on links \( j, k \), directed from \( j \) to \( k \); at time \( t = 0 \), the vector is denoted by \( a_{jk} \)

\( v_j, v_k \) = vector velocities of axes of gears on links \( j, k \)

The linear displacement equation associated with the \( f \)-circuit containing geared edge \( jk \) in complex-number form is:

\[
v_k - v_j = i \omega_k a_{jk} e^{iatu}
\]  

(5)

where \( i^2 = -1 \).

There are \( (n-1)F \) such equations for the epicyclic chain. Since every vertex is either a transfer vertex, or incident at a geared edge, all vertices are included as subscripts in the set (5).

As in the case of rotational displacements, it is not necessary to draw the graph—all that is needed is the identification of the gear carrier (transfer vertex) associated with each gear mesh; equation (5) is determined directly from the graph in a purely formal manner.

Example. For the graph of the epicyclic chain shown in Fig. 3(b), vertex 1 is the transfer vertex for the three \( f \)-circuits. Hence

\[
\begin{align*}
v_1 - v_4 &= i \omega_1 a_{41} e^{iatu} \\
v_1 - v_3 &= i \omega_1 a_{31} e^{iatu} \\
v_2 - v_3 &= i \omega_2 a_{31} e^{iatu}
\end{align*}
\]  

(5a)

If the vertex 4 represents the fixed link, for example, \( v_4 = 0 \), \( v_1 = i \omega_4 a_{41} e^{iatu} \) and \( v_2 = v_3 = 0 \).

Displacement Isomorphism

1 Rotational Isomorphism

We begin with some definitions.

Definition 1: Induced Correspondence. A correspondence between the links of two epicyclic chains can also be applied to the terms of their displacement equations, equations (4) and (5), by noting that a subscript, \( j, k \) in equations (4) and (5) corresponds to link \( j \). The latter correspondence is called an induced correspondence.

Definition 2: Isomorphic Displacement Equations. The displacement equations of two epicyclic chains are said to be isomorphic if there is a 1:1 correspondence between their links, which induces a 1:1 correspondence in their displacement equations.

Epicyclic chains with isomorphic displacement equations will have identical displacement equations if the subscripts of one of them are suitably relabeled.

These definitions may be applied to the rotational or to the linear displacement equations, either separately or jointly.

Definition 3: Rotation Graph. The rotation graph of an epicyclic chain is defined as the graph obtained from the graph of the epicyclic chain by deleting the turning-pair edges and the transfer vertices and labeling each geared edge with the symbol for the associated transfer vertex.
For example, the rotation graph of the epicyclic chain, the graph of which is shown in Fig. 4(b), is shown in Fig. 4(c).

**Definition 4: Isomorphism of Rotation Graphs.** Two rotation graphs are isomorphic if there is a 1:1 correspondence between their vertices and edges, which preserves incidence and labeling.

The rotation graph can be partitioned into $(s-1-F)$ subgraphs, each consisting of a geared edge $(ij)$, labeled according to the associated transfer vertex $k$, say, and the vertices $ij$. The displacement equation corresponding to the subgraph is equation (4). Conversely, given equation (4), we can construct this subgraph. The rotation graph is obtained by joining the subgraphs at common vertices and edges. Hence, there is a 1:1 correspondence between the rotation graph of an epicyclic chain and the rotational displacement equations and we have:

**Theorem 1:** If two epicyclic chains have isomorphic rotation graphs, their rotational displacement equations are isomorphic.

To illustrate this and other results, we make extensive use of the enumeration of geared kinematic chains in Buchsbaum [1] and we shall use the same identification numbers.

**Example.** Epicyclic chain 1400-1-7 has the rotation graph shown in Fig. 4(a); Fig. 4(c) shows the rotation graph of epicyclic chain 2210-1-3 enumerated in Buchsbaum [1]; its graph is shown in Fig. 4(b). By applying the permutation (143) (25) to Fig. 4(c), the two graphs are seen to be isomorphic.

By applying the foregoing to the enumeration in Buchsbaum [1] of single-degree-of-freedom epicyclic chains with up to three gear pairs, we find that there are 9 sets of rotationally nonisomorphic graphs. Referring to Fig. 6 these are as follows:

1. 3000-1-1
2. 2200-1-1, 2200-1-2b
3. 2200-1-3
4. 1400-1-3, 2210-1-1a, b, c
5. 1400-1-4
6. 1400-1-7, 2210-1-2b
7. 2210-1-4a, b, 3020-1-3b
8. 2210-7
9. 3020-1-4b, 2210-6

**2 Linear Isomorphism**

**Definition 5: Displacement Graph.** The displacement graph of an epicyclic chain is defined as the graph obtained from the graph of the epicyclic chain by deleting the turning edges and the transfer vertices, labeling each geared edge with the symbol for the corresponding transfer vertex, and labeling each vertex according to the levels of the edge pair connecting the vertex to the transfer vertex.
vertices in the \( f \)-circuits in which the vertex lies. If the vertex is associated with several levels, each level should be associated with the corresponding \( f \)-circuit.

Fig. 5 illustrates the displacement graph of epicyclic chain 1400-1-7 shown in Fig. 6. It is obtainable by adding the vertex labeling to the rotation graph, Fig. 4(a).

**Definition 6: Isomorphism of Displacement Graphs.** Two displacement graphs are said to be isomorphic if there is a 1:1 correspondence between their vertices and edges, which preserves incidence and labeling of edges and vertices.

The displacement graph can be partitioned into \((n-1-F)\) subgraphs, each of which consists of a geared edge \( jk \), say, labeled according to the associated transfer vertex \( I \), say; the vertices \( j, k \) are labeled according to the edge levels \( jf, kf \). The linear displacement equation corresponding to this subgraph is equation (5). Conversely, given equation (5), we can construct this subgraph; the displacement graph is obtained by joining the subgraphs at common vertices and edges. Hence, there is a 1:1 correspondence between the displacement graph of an epicyclic chain and the linear displacement equations and we have:

**Theorem 2:** If the displacement graphs of two epicyclic chains are isomorphic, their linear displacement equations are isomorphic.

**3 Combined Rotational and Linear Displacement Equations**

Since the displacement graph contains the rotation graph as a subgraph, we have the following:

**Theorem 3:** If the displacement graphs of two epicyclic chains are isomorphic, their rotational and linear displacement equations are isomorphic.

By applying the above to the enumeration in Buchsbaum [1], we can find the graph specifications of all displacement-non-isomorphic single-degree-of-freedom epicyclic chains (actually \( F = 4 \) for the chain and \( F = 1 \) for the drive), with up to 3 gear pairs. There are 17 epicyclic chains. Table 1 lists their graph specification, Fig. 6 shows the graphs, and Fig. 7 the functional schematics.
4 Pseudoisomorphic Graphs

If the displacement graph of an epicyclic chain is known, we may ask what can be concluded about the graph of the epicyclic chain. We can deduce the nature of the graph of the chain by means of the following sequence of steps:

S1 Draw the subgraph, $G$, of the graph of the chain consisting only of the geared edges and their endpoints.

S2 Add the transfer vertices not already in $G$, as isolated vertices, thus obtaining subgraph $H$.

Each vertex obtained in step S1 is associated with the levels of the pair axes of its gears as given in the displacement graph. If the subgraph $H$ contains $n$ vertices having a common level, they must be edge-connected to a single vertex. Hence:

S3 If $n = 2$, the vertex is directly edge-connected to its transfer vertex.

S4 $n > 2$: Each vertex of the subgraph $G$ is associated with one or more transfer vertices of the geared edge incident at the vertex. The transfer vertices are determined from the displacement graph. Let the association of vertex $i$ and transfer vertex $j$ be denoted by the ordered number pair $i, j$. If a vertex is associated with several transfer vertices, a number pair is associated with each transfer vertex. Then the associated transfer vertex of the $n$ vertices of subgraph $H$ at a common level is defined as the vertex common to their ordered number pairs. We remark that not counting loops with only one turning edge there must be at least one associated transfer vertex, for otherwise it would not be possible to connect all $n$ vertices to a single vertex.

<table>
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<th>Graph specification of the 17 displacement nonisomorphic single-degree-of-freedom epicyclic chains with up to 3 gear pairs</th>
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Further, since the number pairs are distinct there cannot be more than one associated transfer vertex.

Hence, we have the following procedure: Select any one of the \( n \) vertices, or the associated transfer vertex, and edge-connect all the vertices directly to the selected vertex. Any of the \( n \) vertices including the associated transfer vertex is admissible, because conditions G1-G8 are not violated. The operation of connecting edges at a common level to one vertex will be called vertex selection.

This completes the construction of the graph of the epicyclic chain.

Hence, if \( n \) exceeds two, there are several graphs corresponding to a given displacement graph.

**Definition 7: Pseudoisomorphism.** Two graphs satisfying conditions G1-G8 are called pseudoisomorphic if they become isomorphic under the single or repeated application of vertex selection in accordance with step \( S4 \).

Hence, we have the following:

**Theorem 4:** If the graphs of two epicyclic chains are pseudoisomorphic, their rotational and linear displacement equations are isomorphic.

5 **Canonical Representation of Epicyclic Chains**

Finally, this suggests a canonical representation of the graph of an epicyclic chain of freedom, one in which \( \sigma_0 \leq 3 \).

We choose as the graph representation the graph in which all vertices of the displacement subgraph \( H \) having a common level are incident at the (uniquely defined) associated transfer vertex. Hence, for the canonical representation, the displacement equation and the graph are in a 1:1 correspondence.

The graphs shown in Fig. 6 are canonical. We conclude also, by the way, that the graph of every epicyclic chain has the basic graph 3000-1-1 of Fig. 6 as a subgraph. Hence, full mobility in the sense of T. H. Davies [15] cannot be obtained when the freedom is greater than 1.

Fig. 6 shows the canonical graph representations of the 17 displacement nonisomorphic epicyclic chains of Table 1.

**Animation of Epicyclic Drives**

The graph of an epicyclic drive contains all the structural information needed to sketch the mechanism, once dimensional and other numerical inputs are available. Animation could proceed, for example, in terms of the following procedure:

1. Represent each level by a point in plane space.
2. Connect the levels by the gear carriers of the drive, in accordance with the graph structure.
3. Draw the pitch circles of each gear centered at the level of the turning pair represented by the incident turning edge.
4. Compute the linear displacements of each pair axis and the angular velocity of each link from equations (4) and (5).

**Dynamical Analysis**

The equations of motion are also determined from the graph. For example, we may introduce elastic compliance and damping by replacing the rigid-body coupling between gears by a coupling involving only elastic compliance and damping. The former might represent shaft compliance and the latter bearing friction and internal hysteresis. The form of the rotational and linear displacement equations derived earlier will not be affected, but we have introduced additional degrees of freedom into the system. The kinetic energy, potential energy, and Rayleigh dissipation function can be found directly from the graph and it is then a purely formal operation to obtain the Lagrangian equations of motion of the system.

An experimental computer program incorporating these concepts has been under development in the Department of Mechanical Engineering at Columbia University.

**Conclusions**

The kinematic structure and motion analysis of epicyclic drives has been considered from the point of view of their graph structure. The investigation illustrates the steps required in a fully automated motion analysis of even a kinematically simple mechanical component.

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**References**